

Generic coarse geometry of leaves

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ABSTRACT. A compact Polish foliated space is considered. Part of this work studies coarsely quasi-isometric invariants of leaves in some residual saturated subset when the foliated space is transitive. In fact, we also use “equi-” versions of this kind of invariants, which means that the definition is satisfied with the same constants by some given set of leaves. For instance, the following properties are proved.

Either all dense leaves without holonomy are equi-coarsely quasi-isometric to each other, or else there exist residually many dense leaves without holonomy such that each of them is coarsely quasi-isometric to meagerly many leaves. Assuming that the foliated space is minimal, the first of the above alternatives holds if and if the leaves without holonomy satisfy a condition called coarse quasi-symmetry.

A similar dichotomy holds for the growth type of the leaves, as well as an analogous characterization of the first alternative in the minimal case, involving a property called growth symmetry. Moreover some classes of growth are shared, either by residually many leaves, or by meagerly many leaves.

If some leaf without holonomy is amenable, then all dense leaves without holonomy are equi-amenable, and, in the minimal case, they satisfy a property called amenable symmetry.

Residually many leaves have the same asymptotic dimension.

If the foliated space is minimal, then any pair of nonempty open sets in the Higson coronas of the leaves with holonomy contain homeomorphic nonempty open subsets.

Another part studies limit sets of leaves at points in the coronas of their compactifications, defined like their usual limit sets at their ends. These sets are nonempty and compact, but they may not be saturated. The following properties are shown.

The limit sets are saturated for compactifications less or equal than the Higson compactification of the leaves. This establishes a relation between the coarse geometry of the leaves and the structure of closed saturated sets.

For any given leaf, its limit set at every point in its Higson corona is the whole space if and only if the foliated space is minimal.

For some dense open subset of point in the Higson corona of any leaf, the corresponding limit sets are minimal sets.

CHAPTER 1

Introduction

Let (X, \mathcal{F}) be a compact Polish foliated space; i.e., this foliated space is compact, Hausdorff and second countable. A regular foliated atlas, \mathcal{U} , for (X, \mathcal{F}) induces a coarse metric, $d_{\mathcal{U}}$, on each leaf L [Hur94]: for $x, y \in L$, $d_{\mathcal{U}}(x, y)$ is the minimum number of \mathcal{U} -plaques whose union is connected and contains x and y . A metric $d_{\mathcal{U}}^*$ on L can be obtained by modifying $d_{\mathcal{U}}$ on the diagonal of $L \times L$, where $d_{\mathcal{U}}^*$ is declared to be zero. By the compactness of X and the regularity of \mathcal{U} , it follows that the coarse quasi-isometry type of the leaves with $d_{\mathcal{U}}^*$ (in the sense of Gromov [Gro93]) is independent of the choice \mathcal{U} . In this way, the leaves are equipped with a coarse quasi-isometry type (of metrics) determined by (X, \mathcal{F}) . The leaves of (X, \mathcal{F}) belong to a class of metric spaces for which the coarse quasi-isometry type equals the coarse type [Roe96], [HR00], [Roe03]. Thus terms like “coarse equivalence” and “coarse type” could be used as well in our statements. Moreover Roe [Roe03] extended to coarse spaces many of the quasi-isometric invariants and properties we use (bounded geometry, growth, amenability, Higson corona and asymptotic dimension).

Let (X, \mathcal{F}) be leafwise differentiable of class C^3 , and g be a C^2 leafwise Riemannian metric on X . Then the coarse quasi-isometry type of the leaves is also represented by the Riemannian distance induced by g . The restrictions of g also define a *differentiable quasi-isometry type* of the leaves, which depends only on (X, \mathcal{F}) .

The determination of the coarse quasi-isometry type of the leaves of a foliated space obtained in the present work is in fact slightly stronger than stated above. A regular foliated atlas \mathcal{U} for (X, \mathcal{F}) determines a set of metric spaces $\{(L, d_{\mathcal{U}}^*)\}$, where $\{L\}$ is the set of leaves of (X, \mathcal{F}) . If \mathcal{U} and \mathcal{V} are regular foliated atlases for (X, \mathcal{F}) , then the set of metric spaces $\{(L, d_{\mathcal{U}}^*)\}$ is *equi-coarsely quasi-isometric* to the set of metric spaces $\{(L, d_{\mathcal{V}}^*)\}$ in the sense that all the coarse quasi-isometries between each pair of metric spaces $(L, d_{\mathcal{U}}^*)$ and $(L, d_{\mathcal{V}}^*)$ share common distortion constants (a coarsely quasi-isometric version of equicontinuity). This kind of terminology will be used with several concepts of metric spaces, with a similar meaning, when their definition involves constants.

The first goal of this work is to study the coarse quasi-isometry properties of the *generic* leaf of (X, \mathcal{F}) , that is, to study what quasi-isometry properties are generic in the topological sense that there is a residual saturated subset of the foliated space all whose leaves share that quasi-isometry property. Thus we will focus on the generic coarse properties of the leaves. For the existence of interesting generic properties of the leaves, it will be often required that the foliated space be transitive (some leaf is dense) or even minimal (all leaves are dense); these conditions, transitivity and minimality, are the topological counterparts of ergodicity.

This part may be thus be placed in the area of Ghys [Ghy95], and Cantwell and Conlon [CC98], who have studied the topology of the generic leaf, where

“generic” was used in a measure theoretic sense by Ghys, and in a topological sense by Cantwel and Conlon. Indeed, it was Ghys who raised the problem of studying the quasi-isometry type of the generic leaf.

For a Polish foliated space (X, \mathcal{F}) , the following notation will be standing for the rest of this work. The leaf of (X, \mathcal{F}) that contains $x \in X$ is denoted by L_x . The subset of X consisting of the union of all leaves that have no holonomy is denoted by X_0 , and the subset of X consisting of all the leaves that are dense and have no holonomy is denoted by $X_{0,d}$. It is well known that X_0 is a residual subset of X , and that, if (X, \mathcal{F}) is transitive, then $X_{0,d}$ is also residual in X . In most of the results of this work, the generic leaf is a leaf in X_0 or in $X_{0,d}$.

The main theorems are stated presently.

THEOREM 1.1. *Let (X, \mathcal{F}) be a compact Polish foliated space. The equivalence relation “ $x \sim y$ if and only if the leaf L_x is coarsely quasi-isometric to the leaf L_y ” has a Borel relation set in $X_0 \times X_0$, and has a Baire relation set in $X \times X$; in particular, it has Borel equivalence classes in X_0 , and Baire equivalence classes in X .*

THEOREM 1.2. *Let (X, \mathcal{F}) be a transitive compact Polish foliated space. Then the following dichotomy holds:*

- (i) *Either all leaves in $X_{0,d}$ are equi-coarsely quasi-isometric to each other; or else*
- (ii) *there exist residually many leaves in $X_{0,d}$ such that each of them is coarsely quasi-isometric to meagerly many leaves; in particular, in this case, there are uncountably many coarse quasi-isometry types of leaves in $X_{0,d}$.*

In Theorem 1.2, the alternative (i) means that there exists a generic quasi-isometry type of leaves. This holds for instance in the case of equicontinuous foliated spaces under some mild conditions [ALC09].

The next theorem characterizes the alternative (i) of Theorem 1.2 by using a property of metric spaces called *coarse quasi-symmetry*. A metric space is *coarsely quasi-symmetric* if it admits a set of equi-coarsely quasi-isometric transformations that is *transitive* in the sense that, for every pair of points of the metric space, there is a coarse quasi-isometry in that set mapping one point onto the other. (Equi-) coarse quasi-symmetry is invariant by (equi-) coarse quasi isometries.

THEOREM 1.3. *The following properties hold for a compact Polish foliated space (X, \mathcal{F}) :*

- (i) *Suppose that (X, \mathcal{F}) is transitive. If one leaf is coarsely quasi-symmetric, then the alternative (i) of Theorem 1.2 holds.*
- (ii) *Suppose that (X, \mathcal{F}) is minimal. If the alternative (i) of Theorem 1.2 holds, then all leaves in X_0 are equi-coarsely quasi-symmetric.*

Roe [Roe93, Proposition 2.25] notes that the number of ends is a coarse invariant for some class of metric spaces, which includes the leaves of X . But indeed the end space is a coarse invariant of each leaf; this is proved by introducing the coarse end space of a metric space, and showing that it equals the usual end space under certain conditions.

The space of ends of the generic leaf plays an important role in the indicated studies of its topology; in particular, a slight simplification of [CC98, Theorem A] states that, if (X, \mathcal{F}) is minimal, then residually many leaves without holonomy

have zero, one, two or a Cantor space of ends, simultaneously. Since the end space is a coarse invariant, we directly get the following slight improvement of that result when alternative (i) of Theorem 1.2 holds.

COROLLARY 1.4. *Suppose that (X, \mathcal{F}) is minimal and satisfies the alternative (i) of Theorem 1.2. Then all leaves in X_0 have zero, one, two or a Cantor space of ends, simultaneously.*

Another proof of Corollary 1.4 is also given, showing that, under certain mild conditions, any coarsely quasi-symmetric metric space has zero, one, two or a Cantor space of coarse ends (Theorem 2.146).

Corollary 1.4 is useful to give examples of minimal compact Polish foliated spaces satisfying the alternative (ii) of Theorem 1.2. For instance, this alternative is satisfied by the foliated space of Ghys-Kenyon [Ghy00] because it has a leaf without holonomy with four ends. Lozano [LR08] gives variations of the example of Ghys-Kenyon, producing minimal foliated spaces with leaves without holonomy that have any finite number ≥ 3 of ends; they also must satisfy alternative (ii) of Theorem 1.2 by Corollary 1.4.

Blanc [Bla03, Théoreme 1] has shown that, if \mathcal{F} is minimal and residually many leaves have two ends, then all leaves without holonomy are coarsely quasi-isometric to \mathbb{Z} , and the leaves with holonomy are coarsely quasi-isometric to \mathbb{N} and have holonomy group $\mathbb{Z}/2\mathbb{Z}$. This completely describes the coarse quasi-isometry type of the leaves in the case with two ends of Corollary 1.4. In particular, it follows that \mathcal{F} satisfies alternative (ii) of Theorem 1.2 if it is minimal and has a leaf with two ends that is not coarsely quasi-isometric to \mathbb{Z} . For instance, the Ghys-Kenyon foliated space also contains leaves with two ends that are not quasi-isometric to \mathbb{Z} [ACLRMS09].

Observe that Theorems 1.1–1.3 and Corollary 1.4 cannot be directly extended to the leaves with finite holonomy groups by the theorem of Blanc quoted above.

One of the classic coarsely quasi-isometric invariants of a leaf L of a compact Polish foliated space (X, \mathcal{F}) is its *growth type*. It can be defined as the growth type of the function $r \mapsto v_{\mathcal{U}}(x, r)$ ($r > 0$), for any regular foliated atlas \mathcal{U} and $x \in L$, where $v_{\mathcal{U}}(x, r)$ is the number of plaques of \mathcal{U} that meet the $d_{\mathcal{U}}^*$ -ball of center x and radius r —this growth type is independent of \mathcal{U} and x . If the functions $r \mapsto v_{\mathcal{U}}(x, r)$ ($x \in L$) have equi-equivalent growth, then L is called *growth symmetric*. If (X, \mathcal{F}) is C^3 and g is a C^2 leafwise Riemannian metric on X , then the growth type of L equals the usual growth type of the connected Riemannian manifold L with the restriction of g .

Block and Weinberger [BW92] introduced the class of metric spaces of *coarse bounded geometry*, and defined the *growth type* for any metric space in this class; it agrees with the above definition for leaves of \mathcal{F} , which are included in this class. Growth symmetry can be also defined with this generality. (Equi-) growth type and (equi-) growth symmetry are invariant by (equi-) coarse quasi-isometries.

THEOREM 1.5. *Let (X, \mathcal{F}) be a compact Polish foliated space. The equivalence relation “ $x \sim y$ if and only if L_x has the same growth type as L_y ” has a Borel relation set in $X_0 \times X_0$, and has a Baire relation set in $X \times X$; in particular, it has Borel equivalence classes in X_0 , and Baire equivalence classes in X .*

THEOREM 1.6. *Let (X, \mathcal{F}) be a transitive compact Polish foliated space. Then the following dichotomy holds:*

- (i) Either all leaves in $X_{0,d}$ have equi-equivalent growth; or else
- (ii) there exist residually many leaves in $X_{0,d}$ such that the growth type of each of them is comparable with the growth type of meagerly many leaves; in particular, in this second case, there are uncountably many growth types of leaves in $X_{0,d}$.

The alternative (i) of Theorem 1.6 is the case where there is a generic growth type.

THEOREM 1.7. *Let (X, \mathcal{F}) be a compact Polish foliated space. The following properties hold:*

- (i) Suppose that (X, \mathcal{F}) is transitive. If there is a growth symmetric leaf in $X_{0,d}$, then the alternative (i) of Theorem 1.6 holds.
- (ii) Suppose that (X, \mathcal{F}) is minimal. If the alternative (i) of Theorem 1.6 holds, then all leaves in X_0 are equi-growth symmetric.

THEOREM 1.8. *Suppose that (X, \mathcal{F}) is a transitive compact Polish foliated space. Then there are $a_1, a_3 \in [1, \infty]$, $a_2, a_4 \in [0, \infty)$ and $p \geq 1$ such that*

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log v(x, r)}{\log r} &= a_1, & a_2 &\leq \liminf_{r \rightarrow \infty} \frac{\log v(x, r)}{r} \leq p a_2, \\ \liminf_{r \rightarrow \infty} \frac{\log v(x, r)}{\log r} &= a_3, & \limsup_{r \rightarrow \infty} \frac{\log v(x, r)}{r} &= a_4 \end{aligned}$$

for residually many points x in X . Moreover

$$\liminf_{r \rightarrow \infty} \frac{\log v(x, r)}{\log r} \geq a_3, \quad \limsup_{r \rightarrow \infty} \frac{\log v(x, r)}{r} \leq a_4$$

for all $x \in X_{0,d}$.

The conditions used in the following result are defined by requiring that the superior and inferior limits used in Theorem 1.8 be $< \infty$, > 0 or ≤ 0 (see Sections 2.1 and 2.5.1). All of those terms are standard except pseudo-quasi-polynomial, which is introduced here. It is well known that the growth type of all non-compact leaves of \mathcal{F} is at least linear and at most exponential.

COROLLARY 1.9. *Let (X, \mathcal{F}) be a transitive compact Polish foliated space. Then the following sets are either meager or residual in X :*

- (i) the union of leaves in X_0 with polynomial growth;
- (ii) the union of leaves in X_0 with exponential growth;
- (iii) the union of leaves in X_0 with quasi-polynomial growth;
- (iv) the union of leaves in X_0 with quasi-exponential growth; and
- (v) the union of leaves in X_0 with pseudo-quasi-polynomial growth.

Moreover,

- (a) if the set (iii) is residual in X , then it contains $X_{0,d}$; and,
- (b) if one of the sets (iv) or (v) is meager in X , then it does not meet $X_{0,d}$.

Hector [Hec77b] constructed a remarkable example of a C^∞ foliation \mathcal{F} of codimension one on $X = M \times S^1$ (where M is the closed oriented surface of genus two), which is transverse to the factor S^1 , satisfying the following properties:

- For each integer $n \geq 0$, there is exactly one proper leaf of exactly polynomial growth of degree n . Every other leaf has non-polynomial growth and is dense.

- Each non-polynomial growth class of leaves has the cardinality of the continuum.
- It has one growth class of leaves with exponential growth.
- The set of growth classes which are non-polynomial but quasi-polynomial (respectively, non-quasi-polynomial but non-exponential) has the cardinality of the continuum.

Furthermore, it can be easily checked in that example that $X_{0,d} = X_0$ is the union of leaves with non-polynomial growth. By Corollary 1.9, it follows that the union of leaves with quasi-polynomial growth is meager in X , and the union of leaves with quasi-exponential growth is residual in X .

Cantwell and Conlon [CC82] have proved that there is a set G of growth types, containing a continuum of distinct quasi-polynomial but non-polynomial types and a continuum of distinct non-exponential but non-quasipolynomial types, together with the exponential type, such that, for any closed C^∞ 3-manifold M and $\gamma \in G$, there is a C^∞ foliation \mathcal{F} on M with a local minimal set U of locally dense type so that:

- $\overline{U} \setminus U$ is a finite union of totally proper leaves;
- the leaves of \mathcal{F} in U have trivial holonomy and are diffeomorphic one another; and
- each leaf of $\mathcal{F}|_U$ has growth type γ .

Then the foliated space $X = \overline{U}$, with the restriction of \mathcal{F} , satisfies the alternative (i) of Theorem 1.6 because $X_0 = X_{0,d} \supset U$ and all leaves have the same growth type on U .

Let \mathcal{U} be a regular foliated atlas for (X, \mathcal{F}) , and let S be a set of \mathcal{U} -plaques in the same leaf L . The boundary ∂S is the set of \mathcal{U} -plaques in L that meet \mathcal{U} -plaques in S and outside S . It is said that L is \mathcal{U} -Følner when there is a sequence of finite sets S_n of \mathcal{U} -plaques such that $|S_n|/|\partial S_n| \rightarrow 0$ as $n \rightarrow \infty$, where $|S_n|$ is the cardinal of S_n . This condition is independent of the choice of \mathcal{U} , and L is called *amenable* when it is satisfied. If (X, \mathcal{F}) is C^3 and g is a C^2 leafwise Riemannian metric on (X, \mathcal{F}) , then a leaf L is amenable if and only if it is Følner as Riemannian manifold with the restriction of g . *Amenable symmetry* is also introduced for a leaf L ; roughly speaking, it means that the Følner condition is satisfied uniformly close to any point of L , and with the same rate of convergence to zero; the precise statement of this definition is indeed very involved. We will also use a property stronger than equi-amenable symmetry called *joint amenable symmetry*.

The concept of amenability was extended to arbitrary metric spaces of coarse bounded geometry by Block and Weinberger [BW92]. All of the above variants of amenability can be also defined in this setting, becoming (equi-) coarsely quasi-isometric invariants.

THEOREM 1.10. *Let (X, \mathcal{F}) be a compact Polish foliated space. The following properties hold:*

- (i) *If (X, \mathcal{F}) is transitive and some leaf in X_0 is amenable, then all leaves in $X_{0,d}$ are equi-amenable.*
- (ii) *If (X, \mathcal{F}) is minimal and some leaf in X_0 is amenable, then all leaves in X_0 are jointly amenably symmetric.*

Another coarse invariant of a metric space M is its *Higson corona* νM . When M is proper, it is the corona of the *Higson compactification* M^ν , which is defined

by applying the Gelfand-Naimark theorem to the bounded continuous functions on M whose variation vanish at infinity. The Higson corona plays an important role in coarse geometry [Roe93], [Roe03]; indeed, a weak version of the germ of νM in M^ν contains all coarse information of M (Proposition 2.150). The *semi weak homogeneity* used in the following theorem means that, given any pair of nonempty open sets, there is a non-empty open subset of one of them homeomorphic to some open subset of the other one—this is weaker than *weak homogeneity* [Fel91], which means that, for all points x and y , there is a pointed homeomorphism $(U, x) \rightarrow (V, y)$ for some open neighborhoods, U of x and V of y .

THEOREM 1.11. *Let (X, \mathcal{F}) be a compact Polish foliated space. If \mathcal{F} is minimal, then the space $\bigsqcup_L \nu L$, with L running in the set of all leaves in X_0 , is semi weakly homogeneous.*

A numerical coarsely quasi-isometric invariant of metric spaces is their *asymptotic dimension*, introduced by Gromov [Gro93]. It can be defined by using covers by uniformly bounded open sets, in a way dual to the definition of Lebesgue covering dimension (by coarsening the covers instead of refining them). The relevance of this invariant of a metric space is illustrated by a theorem of Dranishnikov, Keesling and Uspenikij [DKU98], stating that $\dim \nu M \leq \text{asdim } M$ for all proper metric space M , which is complemented by another theorem of Dranishnikov [Dra00], stating that $\dim \nu M = \text{asdim } M$ if $\text{asdim } M < \infty$. Moreover Yu [Yu98] related the asymptotic dimension to the Novikov conjecture.

THEOREM 1.12. *Let (X, \mathcal{F}) be a transitive compact Polish foliated space. Then residually many leaves have the same asymptotic dimension.*

The following theorem is a coarsely quasi-isometric version of the “Proposition fondamentale” of Ghys [Ghy95, p. 402], using generic leaves in a topological sense.

THEOREM 1.13. *Suppose that (X, \mathcal{F}) is a minimal compact Polish foliated space, and let B be a Baire subset of X . Then:*

- (i) *either the \mathcal{F} -saturation of B is meager; or else*
- (ii) *the intersections of residually many leaves with B are equi-nets in those leaves.*

Due to the relevance of Ghys’ result in the description of the topology of the generic leaf, we expect that this theorem will be relevant to describe the differentiable quasi-isometry type of generic leaves. We will give a counterexample showing that the measure theoretic version of Theorem 1.13 is false. Thus there may be more difficulties to study the measure theoretic generic differentiable quasi-isometric type of leaves. However weaker measure theoretic versions of Theorems 1.2, 1.6 and 1.8 will be proved, involving an ergodic harmonic measure supported in X_0 [Gar83].

Another goal of this work is to study the *limit sets* $\lim_{\mathbf{e}} L$ of a leaf L at a point \mathbf{e} in the corona of any compactification of L , which is a straightforward generalization of the usual limit set of L , or its \mathbf{e} -limit for any end \mathbf{e} of L . For a general compactification of L , the corresponding limit sets of L are closed in X and nonempty, but they may not be \mathcal{F} -saturated. However we will mainly consider the Higson compactification L^ν , or compactifications $\overline{L} \leq L^\nu$; i.e., id_L has a continuous extension $L^\nu \rightarrow \overline{L}$. Then the following theorem gives a bridge between the coarse geometry of the leaves and the structure of closed saturated subsets.

THEOREM 1.14. *Let (X, \mathcal{F}) be a compact Polish foliated space. Let \overline{L} be a compactification of a leaf L , with corona ∂L . If $\overline{L} \leq L^\nu$, then $\lim_{\mathbf{e}} L$ is \mathcal{F} -saturated for all $\mathbf{e} \in \partial L$.*

As defined by Cantwell and Conlon [CC98], a leaf L of \mathcal{F} is totally recurrent if $\lim_{\mathbf{e}} L = X$ for all end \mathbf{e} of L . They showed that the set of totally recurrent leaves of \mathcal{F} , if non-empty, is residual. Correspondingly, using the Higson corona, L is said to be *Higson recurrent* if $\lim_{\mathbf{e}} L = X$ for all $\mathbf{e} \in \nu L$. Higson recurrence behaves in the following manner.

THEOREM 1.15. *Let (X, \mathcal{F}) be a compact Polish foliated space. A leaf is Higson recurrent if and only if \mathcal{F} is minimal.*

For each \mathcal{F} -minimal set Y and every leaf L , let $\nu_Y L = \{ \mathbf{e} \in \nu L \mid \lim_{\mathbf{e}} L = Y \}$.

THEOREM 1.16. *Let (X, \mathcal{F}) be a compact Polish foliated space. For any leaf L , the space $\bigcup_Y \text{Int}_{\nu L}(\nu_Y L)$, where Y runs in the family of \mathcal{F} -minimal sets, is dense in νL .*

The proofs of the main theorems of this work will be carried out in the context of the holonomy pseudogroup of (X, \mathcal{F}) . For a regular foliated atlas \mathcal{U} for (X, \mathcal{F}) , the set E of the transverse components of the changes of coordinates generate a pseudogroup \mathcal{H} on a space Z , which is called a representative of the *holonomy pseudogroup* of (X, \mathcal{F}) . In turn, E defines a metric d_E on the \mathcal{H} -orbits, where $d_E(x, y)$ is the smallest number of elements of E whose composition is defined at x and maps x to y . Then the \mathcal{F} -leaves with $d_{\mathcal{U}}^*$ are coarsely quasi-isometric to the \mathcal{H} -orbits with d_E . In this way, our main theorems are easy consequences of their versions for pseudogroups. Most of the work is devoted to prove those pseudogroup versions, as well as to develop the needed tools about metric spaces.

In the case of Theorem 1.13, an alternative direct proof is also given because it is very short and conceptually interesting. Also, this second proof is representative of the type of direct proofs that could be given for other results. However the pseudogroup versions of these theorems have their own interest; for instance, they can be directly applied to orbits of finitely generated group actions on compact Polish spaces.

The structure of this work is the following. Chapter 2 recalls the needed concepts and results about coarse quasi-isometries, and gives new concepts and results that will be required in the subsequent chapters. Chapter 3 contains preliminaries about pseudogroups, including the pseudogroup version of Theorem 1.13 and a quasi-isometric version of the Reeb local stability for pseudogroups. Chapter 4 contains proofs of the pseudogroup versions of all other main theorems. In Chapter 5, the needed preliminaries on foliated spaces are recalled, and the main results are obtained from their pseudogroup versions. It also includes a section with the measure theoretic versions of Theorems 1.2, 1.6 and 1.8, and Corollary 1.9, a section showing that the measure theoretic version of Theorem 1.13 fails, and another section with open problems.

CHAPTER 2

Coarse geometry on metric spaces

2.1. Notation, conventions and terminology

Symbols M , M' and M'' will denote metric spaces with attendant metrics d , d' and d'' , respectively; $\{M_i\}_{i \in I}$ and $\{M'_i\}$ will denote classes of metric spaces with the same index $I = \{i\}$. Unless otherwise stated, a subset of a metric space becomes a metric space with the induced metric.

Let $r, s \geq 0$, $x \in M$ and $S, T \subset M$. The open and closed balls in M of center x and radius r are denoted by $B_M(x, r)$ and $\overline{B}_M(x, r)$, respectively; in particular, $\overline{B}_M(x, 0) = \{x\}$, and $B_M(x, 0) = \emptyset$. The *penumbra*¹ of S of radius r is the set

$$\text{Pen}_M(S, r) = \bigcup_{y \in S} \overline{B}_M(y, r) .$$

In particular, $\overline{B}_M(x, r) = \text{Pen}_M(\{x\}, r)$. The terms *open/closed r -ball* and *r -penumbra* will be also used to indicate the radius r . Obviously,

$$(2.1) \quad \text{Pen}_M(S \cap T, r) \subset \text{Pen}_M(S, r) \cap \text{Pen}_M(T, r) .$$

Moreover, by the triangle inequality,

$$(2.2) \quad \text{Pen}_M(\text{Pen}_M(S, r), s) \subset \text{Pen}_M(S, r + s)$$

for all $r, s > 0$, and

$$(2.3) \quad r < s \implies \text{Pen}_M(\overline{S}, r) \subset \overline{\text{Pen}_M(S, r)} \subset \text{Pen}_M(S, s) ,$$

where the first inclusion is an equality if M is proper². The *r -boundary*³ of S is the set

$$\partial_r^M S = \text{Pen}_M(S, r) \cap \text{Pen}_M(M \setminus S, r) ;$$

in particular, $\partial_0^M S = \emptyset$. The notation $B(x, r)$, $\overline{B}(x, r)$, $\text{Pen}(S, r)$ and $\partial_r S$ can be also used if it is clear which metric space is being considered. The inclusion

$$(2.4) \quad \partial_r \text{Pen}(S, s) \subset \partial_{r+s} S$$

can be proved as follows. For each $x \in \partial_r \text{Pen}(S, s)$, there are $y \in \text{Pen}(S, s)$ and $z \in M \setminus \text{Pen}(S, s) \subset M \setminus S$ such that $d(x, y) \leq r$ and $d(x, z) \leq r$. Then there is $y_0 \in S$ such that $d(y, y_0) \leq s$, obtaining $d(x, y_0) \leq r + s$ by the triangle inequality. Thus $x \in \partial_{r+s} S$.

Given non-decreasing functions⁴ $u, v : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, the function u is said to be *dominated* by the function v , written $u \leq v$, if there are $a, b \geq 1$ and $c > 0$ such that

¹This is slightly different from the definition of this concept given in [Roe96].

²Recall that M is called *proper* if its closed balls are compact

³This is also slightly different from the definition of this concept given in [BW92].

⁴The usual definition of growth type uses functions $\mathbb{Z}^+ \rightarrow \mathbb{R}^+$, but functions $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ give rise to an equivalent concept.

$u(r) \leq a v(br)$ for all $r \geq c$. For all $a, b \geq 1$, $c > 0$ and $e \geq 0$, we have

$$(2.5) \quad b' > b \quad \& \quad u(r) \leq a v(br + e) \quad \forall r \geq c \\ \implies u(r) \leq a v(b'r) \quad \forall r \geq \max \left\{ c, \frac{e}{b' - b} \right\}.$$

If $u \preceq v \preceq u$, then u and v represent the same *growth type* or have *equivalent growth*; this is an equivalence relation and “ \preceq ” defines a partial order relation between growth types called *domination*. For a class of pairs of non-decreasing functions $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, *equi-domination* means that all of those pairs satisfy the above condition of domination with the same constants a , b and c . A class of non-decreasing functions $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ will be said to have *equi-equivalent growth* if they equi-dominate one another.

For non-decreasing functions $u, v : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, and constants $a, b \geq 1$ and $c > 0$, if $u(r) \leq a v(br)$ for all $r \geq c$, then

$$(2.6) \quad \limsup_{r \rightarrow \infty} \frac{\log u(r)}{\log r} \leq \limsup_{r \rightarrow \infty} \frac{\log v(r)}{\log r},$$

$$(2.7) \quad \liminf_{r \rightarrow \infty} \frac{\log u(r)}{\log r} \leq \liminf_{r \rightarrow \infty} \frac{\log v(r)}{\log r},$$

$$(2.8) \quad \liminf_{r \rightarrow \infty} \frac{\log u(r)}{r} \leq b \liminf_{r \rightarrow \infty} \frac{\log v(r)}{r},$$

$$(2.9) \quad \limsup_{r \rightarrow \infty} \frac{\log u(r)}{r} \leq b \limsup_{r \rightarrow \infty} \frac{\log v(r)}{r}.$$

Thus it makes sense to say that the growth type of u is:

- *exactly polynomial* of degree $d \in \mathbb{N}$ if it is the growth type of the function $r \mapsto r^d$;
- *polynomial* if it is dominated by a polynomial growth of some exact degree; and
- *exponential* if it is the growth type of the function $r \mapsto e^r$, which is the same as the growth of $r \mapsto a^r$ for any $a > 1$.

Observe that the growth type of u is:

- polynomial if and only if $\limsup_{r \rightarrow \infty} \frac{\log u(r)}{\log r} < \infty$; and
- exponential if and only if $0 < \liminf_{r \rightarrow \infty} \frac{\log u(r)}{r} < \infty$.

It is also said that the growth type of u is:

- *quasi-polynomial*⁵ if $\limsup_{r \rightarrow \infty} \frac{\log u(r)}{r} \leq 0$;
- *quasi-exponential* if $0 < \limsup_{r \rightarrow \infty} \frac{\log u(r)}{r} < \infty$; and
- *pseudo-quasi-polynomial* if $\liminf_{r \rightarrow \infty} \frac{\log u(r)}{\log r} < \infty$.

2.2. Coarse quasi-isometries

2.2.1. Coarse quasi-isometries. A map $f : M \rightarrow M'$ is *Lipschitz* if there is some $C > 0$ such that $d'(f(x), f(x')) \leq C d(x, y)$ for all $x, y \in M$. Such a C will be called a *Lipschitz distortion* of f . The map f is called *bi-Lipschitz* when there is $C \geq 1$ such that

$$\frac{1}{C} d(x, y) \leq d'(f(x), f(x')) \leq C d(x, y)$$

⁵This property is sometimes called *subexponential*.

for all $x, y \in M$. In this case, C will be called a *bi-Lipschitz distortion* of f . The term C -(bi)-Lipschitz may be also used for a (bi-)Lipschitz map with (bi-)Lipschitz distortion C . A 1-Lipschitz map is called *non-expanding*. A class of (bi-)Lipschitz maps is called *equi-(bi-)Lipschitz* when they have some common (bi-)Lipschitz distortion. Two metrics d_1 and d_2 on a set S are called *Lipschitz equivalent* if the identity map $(S, d_1) \rightarrow (S, d_2)$ is bi-Lipschitz. Let $\{S_i\}$ be a class of sets each endowed with two metrics $d_{i,1}$ and $d_{i,2}$; the class $\{d_{i,1}\}$ is *equi-Lipschitz equivalent* to $\{d_{i,2}\}$ if, for all i , the identity maps $(S_i, d_{i,1}) \rightarrow (S_i, d_{i,2})$ are equi-bi-Lipschitz.

REMARK 2.1. Any bi-Lipschitz map is injective. If $f : M \rightarrow M'$ and $f' : M' \rightarrow M''$ are (bi-)Lipschitz maps with respective (bi-)Lipschitz distortions C and C' , then $f'f : M \rightarrow M''$ is a (bi-)Lipschitz map with (bi-)Lipschitz distortion CC' .

A subset A of M is called a *net* in M if there is $K \geq 0$ such that $\text{Pen}_M(A, K) = M$. A subset A of M is said to be *separated* if there is $\delta > 0$ such that $d(x, y) > \delta$ for every $x \neq y$ in A . The terms K -net⁶ and δ -separated will be also used. If $\{M_i\}$ is a class of metric spaces, a class of subsets $\{A_i \subset M_i\}$ is called an *equi-net* if there is $K \geq 0$ such that every A_i is a K -net in M_i .

REMARK 2.2. If A is a K -net in M , then it is a K -net in any subset of M that contains A . By the triangle inequality, if A_1 is a K_1 -net in M , and A_2 is a K_2 -net in A_1 , then A_2 is a $(K_1 + K_2)$ -net in M . If $f : M \rightarrow M'$ is a bi-Lipschitz bijection with bi-Lipschitz distortion C , and A is a K -net in M , then $f(A)$ is a CK -net in M' .

LEMMA 2.3 (Álvarez-Candel [ALC11, Lemma 2.1]). *Let $K > 0$ and $x_0 \in M$. There is some K -separated K -net A of M so that $x_0 \in A$.*

REMARK 2.4. In [ALC11, Lemma 2.1], it is not explicitly stated that $x_0 \in A$. In that proof, A is a maximal element of the family of K -separated subsets of M . But the proof works as well using the family of K -separated subsets of M that contain x_0 , obtaining Lemma 2.3. In fact, it can be similarly proved that there is some K -separated K -net containing any given K -separated subset.

DEFINITION 2.5 (Gromov [Gro93]). A *coarse quasi-isometry* of M to M' is a bi-Lipschitz bijection $f : A \rightarrow A'$, where A and A' are nets in M and M' , respectively; in this case, M and M' are said to have the same *coarse quasi-isometry type* or to be *coarsely quasi-isometric*. If A and A' are K -nets, and C is a bi-Lipschitz distortion of f , then the pair (K, C) is called a *coarse distortion* of f ; the term (K, C) -coarse quasi-isometry may be also used in this case. It is said that a map $M \rightarrow M'$ induces a coarse quasi-isometry when its restriction to some subsets of M and M' is a coarse quasi-isometry of M to M' . A coarse quasi-isometry of M to itself will be called a *coarsely quasi-isometric transformation* of M .

REMARK 2.6. If $f : A \rightarrow A'$ is a (K, C) -coarse quasi-isometry of M to M' , then $f^{-1} : A' \rightarrow A$ is a (K, C) -coarse quasi-isometry of M' to M . If moreover $g : B' \rightarrow B''$ is a (K, C) -coarse quasi-isometry of M' to M'' , and $A' \subset B'$, then, using Remarks 2.1 and 2.2, it is easy to check that $gf : A \rightarrow g(A')$ is a coarse quasi-isometry of M to M'' with coarse distortion $(K + CK, C^2)$.

⁶The definition of K -net is slightly different from the definition used in [ALC11]. Our arguments become simpler in this way.

DEFINITION 2.7. Let f_i be a coarse quasi-isometry of each M_i to M'_i . If all of them have a common coarse distortion, then $\{f_i\}$ is called a family of *equi-coarse quasi-isometries*. In this case, $\{M_i\}$ and $\{M'_i\}$ are called *equi-coarsely quasi-isometric*.

DEFINITION 2.8. Two coarse quasi-isometries $f : A \rightarrow A'$ and $g : B \rightarrow B'$ of M to M' are said to be *close* if there are $r, s \geq 0$ such that, for all $x \in A$, there is some $y \in B$ with $d(x, y) \leq r$ and $d'(f(x), g(y)) \leq s$. (In this case, it may be also said that f and g are (r, s) -close, or that f is (r, s) -close to g .)

PROPOSITION 2.9. “Being close” is an equivalence relation on the set of coarse quasi-isometries of M to M' .

PROOF. The relation “Being close” is obviously reflexive. To prove that it is symmetric, let $f : A \rightarrow A'$ and $g : B \rightarrow B'$ be coarse quasi-isometries of M to M' such that f is (r, s) -close to g . Let (K, C) be a coarse distortion for g . For any $y \in B$, there is some $x \in A$ such that $d(x, y) \leq K$. Then there is some $y' \in B$ such that $d(x, y') \leq r$ and $d'(f(x), g(y')) \leq s$. It follows that

$$\begin{aligned} d'(f(x), g(y)) &\leq d'(f(x), g(y')) + d'(g(y'), g(y)) \leq s + C d(y', y) \\ &\leq s + C(d(y', x) + d(x, y)) \leq s + C(r + K), \end{aligned}$$

obtaining that g is $(K, s + C(r + K))$ -close to f .

To prove that the relation “Being close” is transitive, let f and g be as above, and let $h : D \rightarrow D'$ be a coarse quasi-isometry of M to M' that is (t, u) -close to g . By the triangle inequality, it easily follows that f is $(r + t, s + u)$ -close to h . \square

2.2.2. Coarse composites. Let $f : A \rightarrow A'_1$ and $f' : A'_2 \rightarrow A''$ be coarse quasi-isometries of M to M' and of M' to M'' , respectively, and let (K, C) be a coarse distortion for both. The following definition makes sense by Remark 2.6.

DEFINITION 2.10. A *coarse composite* of f and f' is any coarse quasi-isometry of M to M'' that is close to the composite $g'g$, where g (respectively, g') is a coarse quasi-isometry of M to M' (respectively, of M' to M'') close to f (respectively, f') such that $\text{im } g \subset \text{dom } g'$.

PROPOSITION 2.11. Every two coarse composites of f and f' are close.

PROOF. Let g and h be coarse quasi-isometries of M to M' close to f , and let g' and h' be coarse quasi-isometries of M' to M'' close to f' . Suppose that $\text{im } g \subset \text{dom } g'$ and $\text{im } h \subset \text{dom } h'$. Then the composites $g'g : \text{dom } g \rightarrow g'(\text{im } g)$ and $h'h : \text{dom } h \rightarrow h'(\text{im } h)$ are coarse quasi-isometries of M to M'' (Remark 2.6). By Proposition 2.9, there are $r, s, t, u \geq 0$ such that g is (r, s) -close to h , and g' is (t, u) -close to h' . Thus, for each $x \in \text{dom } g$, there is some $y \in \text{dom } h$ so that $d(x, y) \leq r$ and $d(g(x), h(y)) \leq s$. Then there is some $z' \in \text{dom } h'$ such that $d'(g(x), z') \leq t$ and $d''(g'g(x), h'(z')) \leq u$. Let C be a bi-Lipschitz distortion of h' . We get

$$\begin{aligned} d''(g'g(x), h'h(y)) &\leq d''(g'g(x), h'(z')) + d''(h'(z'), h'h(y)) \\ &\leq u + C d'(z', h(y)) \leq u + C(d'(z', g(x)) + d'(g(x), h(y))) \leq u + C(t + s). \end{aligned}$$

This shows that $g'g$ is $(r, u + C(t + s))$ -close to $h'h$. \square

The existence of coarse composites is guaranteed by the following result.

PROPOSITION 2.12. *There is a coarse composite $g : B \rightarrow B''$ of f and f' with coarse distortion $(K(5C+1), 5C^2)$ such that B is a $5KC$ -net of A , B'' is a $3KC$ -net of A'' , and $d'(f(x), f'^{-1}g(x)) \leq 2K$ for all $x \in B$. Furthermore, if $x_1 \in A$ and $x'_2 \in A'_2$ are given so that $d'(f(x_1), x'_2) \leq 2K$, then g can be chosen such that $x_1 \in B$, $f'(x'_2) \in B''$ and $g(x_1) = f'(x'_2)$.*

The following lemma will be used to prove Proposition 2.12.

LEMMA 2.13. *For $K > 0$, let A_1 and A_2 be K -nets of M . Then there is a $(6K, 5)$ -coarsely quasi-isometric transformation $h : B_1 \rightarrow B_2$ of M such that B_1 is a $5K$ -net of A_1 , B_2 is a $3K$ -net of A_2 , and $d(x, h(x)) \leq 2K$ for all $x \in B_1$. Moreover, if $x_1 \in A_1$ and $x_2 \in A_2$ are given so that $d(x_1, x_2) \leq 2K$, then h can be chosen so that $x_1 \in B_1$, $x_2 \in B_2$ and $h(x_1) = x_2$.*

PROOF. By Lemma 2.3, A_k has some K -separated K -net A'_k for $k \in \{1, 2\}$. Then A'_k is a $2K$ -net of M because A_k is K -net of M (Remark 2.2). So, for each $x \in A'_1$, there is some point $h(x) \in A'_2$ such that $d(x, h(x)) \leq 2K$. A map $h : A'_1 \rightarrow A'_2$ is defined in this way, and let B_2 denote its image. Choose some point $g(y) \in h^{-1}(y)$ for each $y \in B_2$, defining a map $g : B_2 \rightarrow A'_1$, whose image is denoted by B_1 . Then the restriction $h : B_1 \rightarrow B_2$ is bijective with inverse equal to the restriction $g : B_2 \rightarrow B_1$.

According to Lemma 2.3, given points $x_k \in A_k$ with $d(x_1, x_2) \leq 2K$, we can take A'_k so that $x_k \in A'_k$, and we can choose $h(x_1) = x_2$ and $g(x_2) = x_1$, obtaining that $x_k \in B_k$.

For each $z \in A_1$, there is $x \in A'_1$ with $d(z, x) \leq K$. Then $h(x) \in B_2$, $gh(x) \in B_1$, and

$$\begin{aligned} d(z, h(x)) &\leq d(z, x) + d(x, h(x)) \leq 3K, \\ d(z, gh(x)) &\leq d(z, h(x)) + d(h(x), gh(x)) \leq 5K. \end{aligned}$$

Thus B_1 is a $5K$ -net of A_1 , and B_2 is a $3K$ -net of A_2 . It follows that B_1 and B_2 are $6K$ -nets of M because A_1 and A_2 are K -nets of M (Remark 2.2).

Since B_1 and B_2 are K -separated and $h : B_1 \rightarrow B_2$ is bijective, for $x \neq y$ in B_1 , we have

$$\begin{aligned} d(h(x), h(y)) &\leq d(h(x), x) + d(x, y) + d(y, h(y)) \\ &\leq d(x, y) + 4K < 5d(x, y), \\ d(x, y) &\leq d(x, h(x)) + d(h(x), h(y)) + d(h(y), y) \\ &\leq d(h(x), h(y)) + 4K < 5d(h(x), h(y)). \quad \square \end{aligned}$$

PROOF OF PROPOSITION 2.12. By Lemma 2.13, there is a $(6K, 5)$ -coarsely quasi-isometric transformation $h : B'_1 \rightarrow B'_2$ of M' such that B'_1 is a $5K$ -net of A'_1 , B'_2 is a $3K$ -net of A'_2 , and $d'(x', h(x')) \leq 2K$ for all $x' \in B'_1$. By Remark 2.2, $B = f^{-1}(B'_1)$ is a $5KC$ -net of A , and $B'' = f'(B'_2)$ is a $3KC$ -net of A'' . Thus, by Remark 2.2, B and B' are $K(5C+1)$ -nets in M and M' because A and A'' are K -nets in M and M'' , respectively. Moreover the composite

$$B \xrightarrow{f} B'_1 \xrightarrow{h} B'_2 \xrightarrow{f'} B''$$

is a $5C^2$ -bi-Lipschitz bijection (Remark 2.1), denoted by $g : B \rightarrow B''$, which satisfies

$$d'(f(x), f'^{-1}g(x)) = d'(f(x), hf(x)) \leq 2K$$

for each $x \in B$.

Observe that the coarse quasi-isometry $f : B \rightarrow B'_1$ is $(0, 0)$ -close to $f : A \rightarrow A'_1$ because $B \subset A$.

It will be now shown that the coarse quasi-isometry $f'h : B'_1 \rightarrow B''$ is close to $f' : A'_2 \rightarrow A''$. For each $x' \in B'_1$ there is $y' \in A'_2$ such that $d'(x', y') \leq K$ because A'_2 is a K -net in M' . Furthermore

$$d''(f'h(x'), f'(y')) \leq C d'(h(x'), y') \leq C(d'(h(x'), x') + d'(x', y')) \leq 3KC,$$

obtaining that $f'h : B'_1 \rightarrow B''$ is $(K, 3KC)$ -close to $f' : A'_2 \rightarrow A''$.

Fix $x_1 \in A$ and $x'_2 \in A'_2$ so that $d'(f(x_1), x'_2) \leq 2K$. By Lemma 2.13, we can choose h such that $f(x_1) \in B'_1$, $x'_2 \in B'_2$ and $hf(x_1) = x'_2$. Hence $x_1 \in B$, $f'(x'_2) \in B''$ and $g(x_1) = f'(x'_2)$. \square

According to Propositions 2.9, 2.12 and 2.11, the closeness classes of coarse quasi-isometries between metric spaces form a category of isomorphisms with the operation induced by coarse composite. The following direct consequence is well known.

COROLLARY 2.14. *“Being coarsely quasi-isometric” is an equivalence relation on metric spaces.*

2.2.3. A coarsely quasi-isometric version of Arzela-Ascoli theorem.

The following proposition will be useful to produce coarse quasi-isometries. It is a version of the Arzela-Ascoli theorem for coarse quasi-isometries.

PROPOSITION 2.15. *Let $\{F_n \subset M\}$ and $\{F'_n \subset M'\}$ be increasing sequences of finite subsets such that $\bigcup_n F_n$ and $\bigcup_n F'_n$ are L -nets in M and M' , respectively. For each n , let f_n be a (K, C) -coarse quasi-isometry of F_n to F'_n , such that $f_n(F_m \cap \text{dom } f_n) = F'_m \cap \text{im } f_n$ if $m < n$. Then there is a $(K+L, C)$ -coarse quasi-isometry g of M to M' , which is the combination of restrictions $f_{n_m} : F_m \cap \text{dom } f_{n_m} \rightarrow F'_m \cap \text{im } f_{n_m}$ for some subsequence f_{n_m} with $n_m \geq m$.*

PROOF. For each m , let \mathcal{C}_m be the set of restrictions $f_n : F_m \cap \text{dom } f_n \rightarrow F'_m \cap \text{im } f_n$ for all $n \geq m$. Define a graph structure on $\mathcal{C} = \bigsqcup_m \mathcal{C}_m$ by placing an edge between each $f \in \mathcal{C}_{m+1}$ and its restriction $F_m \cap \text{dom } f \rightarrow F'_m \cap \text{im } f$, which is well defined and belongs to \mathcal{C}_m ; such a \mathcal{C} is an infinite tree. Each \mathcal{C}_m is finite because so are the sets F_m and F'_m , and thus each vertex of \mathcal{C} meets a finite number of edges. Therefore \mathcal{C} contains an infinite ray with vertices $g_m \in \mathcal{C}_m$; every g_m is a restriction $f_{n_m} : F_m \cap \text{dom } f_{n_m} \rightarrow F'_m \cap \text{im } f_{n_m}$ with $n_m \geq m$. All maps g_m can be combined to define a map $g : \bigcup_m \text{dom } g_m \rightarrow \bigcup_m \text{im } g_m$, which is a (K, C) -coarse quasi-isometry of $\bigcup_n F_n$ to $\bigcup_n F'_n$. Since $\bigcup_n F_n$ and $\bigcup_n F'_n$ are L -nets in M and M' , respectively, this g is a $(K+L, C)$ -coarse quasi-isometry of M to M' (Remark 2.2). \square

REMARK 2.16. In Proposition 2.15, observe that, if $x \in \bigcap_n \text{dom } f_n$ and $f_n(x) = y$ for all n , then $x \in \text{dom } g$ and $g(x) = y$.

2.3. Some classes of maps between metric spaces

2.3.1. Large scale Lipschitz maps.

DEFINITION 2.17. Two maps, $f, g : S \rightarrow M$, of a set, S , into a metric space, M , are said to be *close*⁷ if there is some $c \geq 0$ such that $d(f(x), g(x)) \leq c$ for all $x \in S$; it may be also said that f and g are *c-close*, or that f is *c-close* to g .

REMARK 2.18. “Being close” is an equivalence relation on the set of maps of a set to a metric space.

DEFINITION 2.19 (Gromov [Gro93]). A map⁸ $\phi : M \rightarrow M'$ is said to be *large scale Lipschitz* if there exist $\lambda > 0$ and $b \geq 0$ such that

$$d'(\phi(x), \phi(y)) \leq \lambda d(x, y) + b$$

for all $x, y \in M$; in this case, the pair (λ, b) is called a *large scale Lipschitz distortion* of ϕ , and ϕ is said to be (λ, b) -*large scale Lipschitz*. The map ϕ is said to be *large scale bi-Lipschitz* if there exist constants $\lambda > 0$ and $b \geq 0$ such that

$$\frac{1}{\lambda}(d(x, y) - b) \leq d'(\phi(x), \phi(y)) \leq \lambda d(x, y) + b$$

for all $x, y \in M$; in this case, the pair (λ, b) is called a *large scale bi-Lipschitz distortion* of ϕ , and ϕ is said to be (λ, b) -*large scale bi-Lipschitz*.

A map $\phi : M \rightarrow M'$ is said to be a *large scale Lipschitz equivalence* if it is large scale Lipschitz and there is another large scale Lipschitz map $\psi : M' \rightarrow M$ so that $\psi\phi$ and $\phi\psi$ are close to id_M and $\text{id}_{M'}$, respectively. In this case, if (λ, b) is a large scale Lipschitz distortion of ϕ and ψ , and $\psi\phi$ and $\phi\psi$ are *c-close* to the identity maps, then (λ, b, c) is called a *large scale Lipschitz equivalence distortion* of ϕ ; it may be also said that ϕ is a (λ, b, c) -*large scale Lipschitz equivalence*. In this case, M and M' are said to be *large scale Lipschitz equivalent*. Two metrics d_1 and d_2 on a set S are called *large scale Lipschitz equivalent* if the identity map $(S, d_1) \rightarrow (S, d_2)$ is large scale bi-Lipschitz.

DEFINITION 2.20. Let $\phi_i : M_i \rightarrow M'_i$ for each i . The class $\{\phi_i\}$ is said to be *equi-large scale (bi-)Lipschitz* if all the maps ϕ_i are large scale (bi-)Lipschitz with a common large scale (bi-)Lipschitz distortion. The class $\{\phi_i\}$ is said to be *equi-large scale Lipschitz equivalences* if the maps ϕ_i are large scale Lipschitz equivalences and have a common large scale Lipschitz equivalence distortion; in this case, $\{M_i\}$ and $\{M'_i\}$ are called *equi-large scale Lipschitz equivalent*. Given a class of sets $\{S_i\}$, and metrics $d_{i,1}$ and $d_{i,2}$ on each S_i , $\{d_{i,1}\}$ is said to be *equi-large scale Lipschitz equivalent* to $\{d_{i,2}\}$ if the identity maps $(S_i, d_{i,1}) \rightarrow (S_i, d_{i,2})$ are equi-large scale bi-Lipschitz.

The qualitative content of the following lemmas is well known, but we keep track of the constants involved.

LEMMA 2.21. *If $\phi : M \rightarrow M'$ is a (λ, b, c) -large scale Lipschitz equivalence, then ϕ is $(\lambda, b + 2c)$ -large scale bi-Lipschitz.*

PROOF. Let $\psi : M' \rightarrow M$ be a (λ, b) -large scale Lipschitz map such that $\psi\phi$ and $\phi\psi$ are *c-close* to id_M and $\text{id}_{M'}$, respectively. Then

$$d(x, y) \leq d(\psi\phi(x), \psi\phi(y)) + 2c \leq \lambda d'(\phi(x), \phi(y)) + b + 2c$$

for all $x, y \in M$ by the triangle inequality. □

⁷This terminology is used in [HR00]. Other terms used to indicate the same property are *coarsely equivalent* [Roe96], *parallel* [Gro93], *bornotopic* [Roe93], and *uniformly close* [BW92].

⁸Continuity is not assumed here.

LEMMA 2.22. *Let $\phi : M \rightarrow M'$ be (λ, b) -large scale bi-Lipschitz. If $\phi(M)$ is a c -net in M' , then ϕ is a $(\lambda, b + 2\lambda c, \max\{b, c\})$ -large scale Lipschitz equivalence.*

PROOF. Because $\phi(M)$ is a c -net in M' , a map $\psi : M' \rightarrow M$ can be constructed by choosing, for each $x' \in M'$, one point $\psi(x') \in M$ such that $d'(x', \phi\psi(x')) \leq c$, and furthermore so that, if $x' \in \phi(M)$, then $\phi\psi(x') = x'$; i.e., $\phi\psi\phi = \phi$.

Then, for all $x', y' \in M'$ and $x \in M$,

$$\begin{aligned} d(x, \psi\phi(x)) &\leq \lambda d'(\phi(x), \phi\psi\phi(x)) + b = \lambda d'(\phi(x), \phi(x)) + b = b, \\ d'(x', y') &\leq d'(x', \phi\psi(x')) + d'(\phi\psi(x'), \phi\psi(y')) + d'(\phi\psi(y'), y') \\ &\leq \lambda d(\psi(x'), \psi(y')) + b + 2c, \\ d(\psi(x'), \psi(y')) &\leq \lambda d'(\phi\psi(x'), \phi\psi(y')) + b \\ &\leq \lambda(d'(\phi\psi(x'), x') + d'(x', y') + d'(y', \phi\psi(y'))) + b \\ &\leq \lambda d'(x', y') + b + 2\lambda c. \quad \square \end{aligned}$$

REMARK 2.23. According to Lemmas 2.21 and 2.22, (equi-) large scale Lipschitz equivalences are just (equi-) large scale bi-Lipschitz maps whose images are (equi-) nets.

LEMMA 2.24. *Let $\phi : M \rightarrow M'$ and $\phi' : M' \rightarrow M''$ be maps. The following properties hold:*

- (i) *If ϕ and ϕ' are (λ, b) -large scale Lipschitz, then $\phi'\phi$ is $(\lambda^2, \lambda b + b)$ -large scale Lipschitz.*
- (ii) *If ϕ and ϕ' are (λ, b, c) -large scale Lipschitz equivalences, then $\phi'\phi$ is a $(\lambda^2, \lambda b + b, 2c)$ -large scale Lipschitz equivalence.*

PROOF. Property (i) is true because, for all $x, y \in M$,

$$d''(\phi'\phi(x), \phi'\phi(y)) \leq \lambda d'(\phi(x), \phi(y)) + b \leq \lambda^2 d(x, y) + \lambda b + b.$$

To prove (ii), take (λ, b) -large scale Lipschitz maps $\psi : M' \rightarrow M$ and $\psi' : M'' \rightarrow M'$ such that $\psi\phi$ is c -close to id_M , $\phi\psi$ and $\psi'\phi'$ are c -close to $\text{id}_{M'}$, and $\phi'\psi'$ is c -close to $\text{id}_{M''}$. By (i), $\phi'\phi$ and $\psi\psi'$ are $(\lambda^2, \lambda b + b)$ -large scale Lipschitz. Moreover

$$d(\psi\psi'\phi'\phi(x), x) \leq d(\psi\psi'\phi'\phi(x), \phi'\phi(x)) + d(\phi'\phi(x), x) \leq 2c$$

for all $x \in M$, obtaining that $\psi\psi'\phi'\phi$ is $2c$ -close to id_M . Similarly, $\phi'\phi\psi\psi'$ is $2c$ -close to $\text{id}_{M''}$. \square

LEMMA 2.25. *Let $\phi, \psi : M \rightarrow M'$ and $\phi', \psi' : M' \rightarrow M''$ be (λ, b) -large scale Lipschitz maps. If ϕ and ϕ' are R -close to ψ and ψ' , respectively, then $\phi'\phi$ is $(\lambda R + b + R)$ -close to $\psi'\psi$.*

PROOF. For all $x \in M$,

$$\begin{aligned} d''(\phi'\phi(x), \psi'\psi(x)) &\leq d''(\phi'\phi(x), \psi'\phi(x)) + d''(\psi'\phi(x), \psi'\psi(x)) \\ &\leq R + \lambda d'(\phi(x), \psi(x)) + b \leq \lambda R + b + R. \quad \square \end{aligned}$$

By Lemmas 2.24 and 2.25, the closeness classes of large scale Lipschitz maps between metric spaces form a category, whose isomorphisms are the classes represented by large scale Lipschitz equivalences.

It is well known that two metric spaces are coarsely quasi-isometric if and only if they are isomorphic in the category whose objects are metric spaces and whose

morphisms are closeness equivalence classes of large scale Lipschitz maps. This is part of the content of the following two results, where the constants involved are specially analyzed.

PROPOSITION 2.26 (Álvarez-Candel [ALC11, Proposition 2.2]). *Every (K, C) -coarse quasi-isometry $f : A \rightarrow A'$ of M to M' is induced by a $(C, 2CK, K)$ -large scale Lipschitz equivalence $\phi : M \rightarrow M'$.*

PROPOSITION 2.27 (Álvarez-Candel [ALC11, Proposition 2.3]). *For each $\varepsilon > 0$ and $x_0 \in M$, every (λ, b, c) -large scale Lipschitz equivalence $\phi : M \rightarrow M'$ induces a (K, C) -coarse quasi-isometry $f : A \rightarrow A'$ of M to M' such that $x_0 \in A$, where*

$$K = c + 2\lambda c + \lambda b + \lambda\varepsilon + b, \quad C = \lambda + \frac{\lambda}{\varepsilon}(2c + b).$$

REMARK 2.28. In [ALC11, Proposition 2.3], it is not explicitly stated that A contains any given point x_0 . But, in the proof of that proposition, the set A is any $(2c + b + \varepsilon)$ -separated $(2c + b + \varepsilon)$ -net of M , and so, using Lemma 2.3, it may be further imposed that $x_0 \in A$.

REMARK 2.29. According to Propositions 2.26 and 2.27, $\{M_i\}$ and $\{M'_i\}$ are equi-coarsely quasi-isometric if and only if they are equi-large scale Lipschitz equivalent.

PROPOSITION 2.30. *Let $\phi, \psi : M \rightarrow M'$ be (λ, b) -large scale Lipschitz equivalences, and let $f : A \rightarrow A'$ and $g : B \rightarrow B'$ be (K, C) -coarse quasi-isometries of M to M' induced by ϕ and ψ , respectively. The following properties hold:*

- (i) *If ϕ is R -close to ψ , then f is $(K, R + \lambda K + b)$ -close to g .*
- (ii) *If f is (r, s) -close to g , then ϕ is $(\lambda(r + 2K) + s + 2b)$ -close to ψ .*

PROOF. (i) For all $x \in A$, there is some $y \in B$ so that $d(x, y) \leq K$. Then

$$\begin{aligned} d'(f(x), g(y)) &= d'(\phi(x), \psi(y)) \leq d'(\phi(x), \psi(x)) + d'(\psi(x), \psi(y)) \\ &\leq R + \lambda d(x, y) + b \leq R + \lambda K + b. \end{aligned}$$

(ii) For any $x \in M$, there is some $y \in A$ such that $d(x, y) \leq K$. Then there is some $z \in B$ so that $d(y, z) \leq r$ and $d'(f(y), g(z)) \leq s$. Hence

$$\begin{aligned} d'(\phi(x), \psi(x)) &\leq d'(\phi(x), \phi(y)) + d'(f(y), g(z)) + d'(\psi(z), \psi(x)) \\ &\leq \lambda d(x, y) + s + \lambda d(z, x) + 2b \leq \lambda K + s + \lambda(d(z, y) + d(y, x)) + 2b \\ &\leq \lambda(r + 2K) + s + 2b. \quad \square \end{aligned}$$

By Propositions 2.26, 2.27 and 2.30, the category whose objects are metric space and whose morphisms are closeness equivalence classes of coarse quasi-isometries between metric spaces can be identified to the subcategory of isomorphisms of the category of closeness classes of large scale Lipschitz maps between metric spaces. Therefore coarse quasi-isometries and large scale Lipschitz equivalences are equivalent concepts. We will often use large scale Lipschitz equivalences in the proofs because they become simpler. However direct proofs for coarse quasi-isometries may produce better constants. This will be indicated in remarks.

PROPOSITION 2.31. *Let $\phi : M \rightarrow M'$, $x_0 \in M$, $x'_0 \in M'$ and $R \geq d'(\phi(x_0), x'_0)$, and let $\bar{\phi} : M \rightarrow M'$ be defined by $\bar{\phi}(x_0) = x'_0$ and $\bar{\phi}(x) = \phi(x)$ if $x \neq x_0$. The following properties hold:*

- (i) If ϕ is (λ, b) -large scale Lipschitz, then $\bar{\phi}$ is $(\lambda, b + R)$ -large scale Lipschitz.
- (ii) If ϕ is a (λ, b, c) -large scale Lipschitz equivalence, then $\bar{\phi}$ is a $(\lambda, \bar{b}, \bar{c})$ -large scale Lipschitz equivalence, where $\bar{b} = b + R$ and $\bar{c} = \lambda R + b + 2c$.

PROOF. Since $\bar{\phi}$ equals ϕ on $M \setminus \{x_0\}$, it is enough to check (i) for x_0 and any $x \neq x_0$ in M :

$$\begin{aligned} d'(\bar{\phi}(x_0), \bar{\phi}(x)) &= d'(x'_0, \phi(x)) \leq d'(x'_0, \phi(x_0)) + d'(\phi(x_0), \phi(x)) \\ &\leq R + \lambda d(x_0, x) + b. \end{aligned}$$

To prove (ii), take a (λ, b) -large scale Lipschitz map $\psi : M' \rightarrow M$ so that $\psi\phi$ and $\phi\psi$ are c -close to id_M and $\text{id}_{M'}$, respectively. Let $\bar{\psi} : M' \rightarrow M$ be defined by $\bar{\psi}(x'_0) = x_0$ and $\bar{\psi}(x') = \psi(x')$ if $x' \neq x'_0$. Then $\bar{\phi}$ and $\bar{\psi}$ are $(\lambda, b + R)$ -large scale Lipschitz by (i). Moreover ϕ and ψ are $(\lambda, b + 2c)$ -large scale bi-Lipschitz by Lemma 2.21. Since $\bar{\phi}$ and $\bar{\psi}$ equal ϕ and ψ on $M \setminus \{x_0\}$ and $M' \setminus \{x'_0\}$, respectively, to check that $\bar{\psi}\bar{\phi}$ is $\lambda R + b + 2c$ -close to id_M , the only non-trivial case is at every point $x \in M$ with $\phi(x) = x'_0$:

$$d(x, \bar{\psi}\bar{\phi}(x)) = d(x, x_0) \leq \lambda d'(\phi(x), \phi(x_0)) + b + 2c \leq \lambda R + b + 2c.$$

Similarly, we get that $\bar{\phi}\bar{\psi}$ is $\lambda R + b + 2c$ -close to $\text{id}_{M'}$. \square

REMARK 2.32. In Proposition 2.31, note that $\bar{\phi}$ is R -close to ϕ .

COROLLARY 2.33. Let $f : A \rightarrow A'$ be a (K, C) -coarse quasi-isometry of M to M' . Let $R > 0$, $x_0 \in M$ and $x'_0 \in M'$ such that there is some $y_0 \in A$ with $d(x_0, y_0) \leq R$ and $d'(x'_0, f(y_0)) \leq R$. Then, for all $\varepsilon > 0$, there is a (\bar{K}, \bar{C}) -coarse quasi-isometry $\bar{f} : B \rightarrow B'$ of M to M' (r, s) -close to f such that $x_0 \in B$, $x'_0 \in B'$ and $f(x_0) = x'_0$, where

$$\begin{aligned} \bar{K} &= \bar{c} + 2C\bar{c} + C\bar{b} + C\varepsilon + \bar{b}, \quad \bar{C} = C + \frac{C}{\varepsilon}(2\bar{c} + \bar{b}), \\ r &= \bar{K}, \quad s = CR + 2CK + R + \bar{C}\bar{K} + \bar{b}. \end{aligned}$$

PROOF. By Proposition 2.26, f is induced by a $(C, 2CK, K)$ -large scale Lipschitz equivalence $\phi : M \rightarrow M'$. We have

$$\begin{aligned} d'(\phi(x_0), x'_0) &\leq d'(\phi(x_0), \phi(y_0)) + d'(\phi(y_0), x'_0) \\ &\leq C d(x_0, y_0) + 2CK + R \leq CR + 2CK + R. \end{aligned}$$

According to Proposition 2.31 and Remark 2.32, ϕ is $(CR + 2CK + R)$ -close to a (C, \bar{b}, \bar{c}) -large scale Lipschitz equivalence $\bar{\phi} : M \rightarrow M'$ with $\bar{\phi}(x_0) = x'_0$. By Propositions 2.27 and 2.30-(i), for each $\varepsilon > 0$, $\bar{\phi}$ induces a (\bar{K}, \bar{C}) -coarse quasi-isometry $\bar{f} : B \rightarrow B'$ of M to M' (\bar{K}, s) -close to f such that $x_0 \in B$. \square

REMARK 2.34. Another version of Corollary 2.33, where

$$\bar{K} = 2C^2R + 2CR + R + K, \quad \bar{C} = 2C + 1, \quad r = s = R,$$

can be proved without passing to large scale Lipschitz equivalences, with more involved arguments. These constants are simpler, but the constants of Corollary 2.33 give the following extra information: we can get \bar{C} as close to C as desired at the expense of increasing \bar{K} (by taking ε large enough).

2.3.2. Coarse and rough maps.

DEFINITION 2.35. A map $f : M \rightarrow M'$ is called:

- *uniformly expansive*⁹ if, for each $r \geq 0$, there is some $s_r \geq 0$ such that

$$d(x, y) \leq r \implies d'(f(x), f(y)) \leq s_r$$

for all $x, y \in M$;

- *metrically proper*¹⁰ if $f^{-1}(B)$ is bounded in M for each bounded subset $B \subset M'$;
- *uniformly metrically proper*¹¹ if, for each $r \geq 0$, there is some $t_r \geq 0$ so that

$$d'(f(x), f(y)) \leq r \implies d(x, y) \leq t_r$$

for all $x, y \in M$;

- *coarse*¹² if it is uniformly expansive and metrically proper; and
- *rough* if it is uniformly expansive and uniformly metrically proper.

If f satisfies the conditions of uniform expansiveness and uniform metric properness with respective mappings $r \mapsto s_r$ and $r \mapsto t_r$ (simply denoted by s_r and t_r), then (s_r, t_r) is called a *rough distortion* of f ; the term (s_r, t_r) -*rough map* may be also used. When $s_r = t_r$, we simply say that s_r is a *rough distortion* of f , or f is an s_r -*rough map*. If f is a coarse map and there is a coarse map $g : M' \rightarrow M$ such that gf and fg are close to id_M and $\text{id}_{M'}$, respectively, then f is called a *coarse equivalence*. If f is an s_r -rough map and there is an s_r -rough map $g : M' \rightarrow M$ such that gf and fg are c -close to id_M and $\text{id}_{M'}$, respectively, then f is called an (s_r, c) -*rough equivalence*, and (s_r, c) is called a *rough equivalence distortion* of f . If there is a coarse (respectively, rough) equivalence $M \rightarrow M'$, then M and M' are *coarsely* (respectively, *roughly*)¹³ *equivalent*. A coarse (respectively, rough) equivalence $M \rightarrow M$ is called a *coarse* (respectively, *rough*) *transformation* of M .

Two metrics d_1 and d_2 on the same set S are called *coarsely* (respectively, *roughly*) *equivalent* if the identity map $(S, d_1) \rightarrow (S, d_2)$ is a coarse (respectively, rough) equivalence. When S is equipped with a coarse (respectively, rough) equivalence class of metrics, it is called a *metric coarse space*¹⁴ (respectively, *rough space*). The metric coarse space and rough space induced by the metric space M is denoted by $[M]$. The condition on a map $M \rightarrow M'$ to be coarse (respectively, rough) depends only on the metric coarse spaces (respectively, rough spaces) $[M]$ and $[M']$. Any composition of coarse (respectively, rough) maps is (respectively, rough); more precisely, if $f : M \rightarrow M'$ and $f' : M' \rightarrow M''$ are s_r -rough maps, then $f'f$ is s_{s_r} -rough. Moreover the composite of coarse/rough maps is compatible with the closeness relation in an obvious sense. So the closeness classes of coarse (respectively, rough) maps between metric coarse spaces (respectively, rough spaces) form a category

⁹This name is taken from [Roe96]. Other terms used to denote the same property are *uniformly bornologous* [Roe93] and *coarsely Lipschitz* [BW92].

¹⁰This term is used in [Roe96].

¹¹This term is used in [Roe96]. Another term used to denote the same property is *effectively proper* [BW92].

¹²This is a particular case of coarse maps between general coarse spaces [Roe96], [HR00].

¹³The term *uniform closeness* is used in [BW92] when two metric spaces are roughly equivalent.

¹⁴This notion of metric coarse space is equivalent to the concept of coarse space induced by a metric [Roe96], [Roe03].

called *metric coarse category*¹⁵ (respectively, *rough category*). Thus rough equivalences are the maps that induce isomorphisms in the rough category. There are interesting differences between the rough category and the metric coarse category, cf. [Roe96], but the following result shows that they have the same isomorphisms.

PROPOSITION 2.36 (Álvarez-Candel [ALC11, Proposition 3.8]). *Any coarse equivalence between metric spaces is uniformly metrically proper. Moreover the definition of uniform metric properness is satisfied with constants that depend only on the constants involved in the definition of coarse equivalence.*

Observe that, if $f : M \rightarrow M'$ is an (s_r, c) -rough equivalence, then $f(M)$ is a c -net in M' .

PROPOSITION 2.37. *If $f : M \rightarrow M'$ is s_r -rough and $f(M)$ is a K -net in M' , then f is a (\bar{s}_r, c) -rough equivalence, where $\bar{s}_r = \max\{s_{r+2K}, s_r + 2K\}$ and $c = \max\{K, s_K\}$.*

PROOF. Let $g : M' \rightarrow M$ be defined by choosing, for each $x' \in M'$, a point $g(x') \in M$ so that $d'(x', fg(x')) \leq K$. Thus fg is K -close to $\text{id}_{M'}$.

For $x \in M$, we have $d'(f(x), fgf(x)) \leq K$, giving $d(x, gf(x)) \leq s_K$. Hence gf is s_K -close to id_M .

For $x', y' \in M'$ with $d'(x', y') \leq r$, we have

$$d(fg(x'), fg(y')) \leq d(fg(x'), x') + d'(x', y') + d'(y', fg(y')) \leq r + 2K,$$

obtaining $d(g(x'), g(y')) \leq s_{r+2K}$. Thus g is s_{r+2K} -uniformly expansive.

Suppose now that $d(g(x'), g(y')) \leq r$. Then

$$d'(x', y') \leq d'(x', fg(x')) + d'(fg(x'), fg(y')) + d'(fg(y'), y') \leq s_r + 2K.$$

So g is also $(s_r + 2K)$ -uniformly metrically proper. \square

The following is a direct consequence of Propositions 2.36 and 2.37.

COROLLARY 2.38. *For a map $f : M \rightarrow M'$, the following conditions are equivalent:*

- (i) f is rough.
- (i) $f : M \rightarrow f(M)$ is a rough equivalence.
- (i) $f : M \rightarrow f(M)$ is a coarse equivalence.

According to Corollary 2.38, coarse maps can be also properly called *coarse embeddings*¹⁶.

PROPOSITION 2.39 (Álvarez-Candel [ALC11, Proposition 3.13]). *The following properties are true:*

- (i) Any (λ, b) -large scale Lipschitz map satisfies the condition of uniform expansiveness with $s_r = \lambda r + b$.
- (ii) Any (λ, b, c) -large scale Lipschitz equivalence is an (s_r, c) -rough equivalence, where $s_r = \lambda r + b + 2c$.

REMARK 2.40. In fact, the proof of Proposition 2.39 is elementary: (i) is obvious, and (ii) follows from Lemma 2.21.

¹⁵This is a subcategory of the coarse category [Roe96], [HR00].

¹⁶This concept is generalized to arbitrary coarse spaces as maps that define a coarse equivalence to their image [Roe03, Section 11.1].

DEFINITION 2.41. A class of maps, $\{f_i : M_i \rightarrow M'_i\}$, is said to be a class of:

- *equi-uniformly expansive* maps if they satisfy the condition of uniform expansiveness with the same mapping $r \mapsto s_r$.
- *equi-metrically proper* maps if they satisfy the condition of metric properness with the same mapping $r \mapsto t_r$.
- *equi-rough* maps (or *equi-coarse embeddings*) if they are rough with a common rough distortion; and
- *equi-rough equivalences* (or *equi-coarse equivalences*) if they are rough equivalences with a common rough equivalence distortion.

It is not possible to define “equi-coarse maps,” but the concept of “equi-rough equivalences” makes sense according to Proposition 2.36. Given a family of sets $\{S_i\}$, and metrics $d_{i,1}$ and $d_{i,2}$ on each S_i , it is said that $\{d_{i,1}\}$ is *equi-roughly equivalent* (or *equi-coarsely equivalent*) to $\{d_{i,2}\}$ if the identity maps $(S_i, d_{i,1}) \rightarrow (S_i, d_{i,2})$ are equi-rough equivalences.

EXAMPLE 2.42. If the metric spaces M_i are bounded, they are equi-coarsely equivalent to the singleton metric space if and only $\sup_i \text{diam } M_i < \infty$.

REMARK 2.43. (i) By Proposition 2.37, families of equi-rough maps whose images are equi-nets are equi-rough equivalences.

- (i) By Proposition 2.39, families of equi-large scale Lipschitz maps are equi-uniformly expansive, and families of equi-large scale Lipschitz equivalences are equi-rough equivalences.

There are rough equivalences that are not large scale Lipschitz equivalences, cf. [ALC11, Example 3.14].

PROPOSITION 2.44. *If there are disjoint unions, $M = \bigcup_{i=0}^{\infty} M_i$ and $M' = \bigcup_{i=0}^{\infty} M'_i$, such that:*

- M_i and M'_i are bounded for all i ;
- $\min_{i < j} d(M_i, M_j) \rightarrow \infty$ and $\min_{i < j} d(M'_i, M'_j) \rightarrow \infty$ as $j \rightarrow \infty$ and
- $\{M_i\}$ are equi-coarsely equivalent to $\{M'_i\}$;

then M is coarsely equivalent to M' .

PROOF. There are sequences $u_i, v_i \rightarrow \infty$ such that:

- $d(M_i, M_j) \geq u_j$ and $d(M'_i, M'_j) \geq v_j$ if $i < j$; and
- $\text{diam}(\bigcup_{i \leq j} M_i) \leq v_j$ and $\text{diam}(\bigcup_{i \leq j} M'_i) \leq v_j$ for all j .

Moreover there are rough equivalences $f_i : M_i \rightarrow M'_i$ with a common rough equivalence distortion (s_r, c) . Thus there are s_r -rough maps $g_i : M'_i \rightarrow M_i$ such that $g_i f_i$ and $f_i g_i$ are c -close to id_{M_i} and $\text{id}_{M'_i}$, respectively. Since the sets M_i are disjoint from each other, the maps f_i can be combined to define a map $f : M \rightarrow M'$. Similarly, the maps g_i can be also combined to define a map $g : M' \rightarrow M$.

Clearly, gf and fg are c -close to id_M and $\text{id}_{M'}$, respectively. Thus it only remains to prove that f and g are rough.

For $r \geq 0$, let

$$\bar{s}_r = \max\{s_r, v_j \mid u_j \leq r\}.$$

Suppose that $d(x, y) \leq r$ for some $x, y \in M$ and $r \geq 0$. If $x, y \in M_i$ for some i , then $d'(f(x), f(y)) \leq s_r \leq \bar{s}_r$. If $x \in M_i$ and $y \in M_j$ for some $i < j$, then $u_j \leq d(x, y) \leq r$, and therefore $d'(f(x), f(y)) \leq v_j \leq \bar{s}_r$.

Now, suppose that $d(f(x), f(y)) \leq r$ for some $x, y \in M$ and $r \geq 0$. If $x, y \in M'_i$ for some i , then $d(x, y) \leq s_r \leq \bar{s}_r$. If $x \in M_i$ and $y \in M_j$ for some $i < j$, then $u_j \leq d'(f(x), f(y)) \leq r$, and therefore $d(x, y) \leq v_j \leq \bar{s}_r$. Thus f is \bar{s}_r -rough. In the same way, we get that g is \bar{s}_r -rough. \square

2.4. Some classes of metric spaces

2.4.1. Graphs. Suppose that M is the set of vertices of a connected graph G , equipped with the metric d defined by setting $d(x, y)$ equal to the minimum number of consecutive edges of M needed to joint x and y (being 0 if $x = y$). When equipped with this metric, M is called the metric space of vertices of the connected graph G . Observe that M (equipped with d) and G determine each other. Since d has values in \mathbb{N} , it will be enough to consider (open, closed) r -balls, r -penumbras, r -boundaries, K -nets and δ -separated sets with $r, K, \delta \in \mathbb{N}$.

EXAMPLE 2.45. Let Γ be a group and let S be a generating set for Γ . The Cayley graph of Γ relative to S , $G = G(\Gamma, S)$, is the graph that has one vertex for each element of Γ and one edge joining γ_1 and γ_2 if either $\gamma_1\gamma_2^{-1} \in S$ or $\gamma_2\gamma_1^{-1} \in S$. The vertex set of G is the set of elements of Γ , and the metric induced on Γ is the word metric relative to S , denoted by d_S . It is well known that if Γ is finitely generated then all those metrics that are induced by finite generating sets for Γ are in the same Lipschitz class.

EXAMPLE 2.46. Let $G = G(\Gamma, S)$ be the Cayley graph of a group Γ relative to a generating set S . The group operation of Γ induces an action on the right of Γ on G , and the metric d_S is invariant under this action: $d_S(\gamma_1\gamma, \gamma_2\gamma) = d_S(\gamma_1, \gamma_2)$.

If Γ_0 is a subgroup of Γ (not necessarily a normal subgroup), the right invariant metric d_S on Γ induces a metric on the homogeneous space of right cosets Γ/Γ_0 .

EXAMPLE 2.47. Let Γ be a group. Let S be a generating set for Γ and let d_S be the right invariant metric induced by S on Γ .

Let Γ act on the left on a space X and denote the action by $x \mapsto \gamma x$. There is a natural bijection between the orbit, $\Gamma(x)$ of a point x , consisting of all the points $\gamma(x)$, $\gamma \in \Gamma$, and the space of right cosets of Γ relative to the subgroup of Γ that fixes x : If Γ_x is the set of $\gamma \in \Gamma$ such that $x\gamma = x$, then the assignment $[\gamma] \in \Gamma/\Gamma_x \mapsto \gamma x$ is independent of the representative of the class $[\gamma]$, for if $\gamma' \in [\gamma]$, then $\gamma^{-1}\gamma'$ fixes x , so $\gamma'x = \gamma x$.

The metric d_S induced by S on Γ in turn induces a metric on the orbit $\Gamma(x)$. This metric is actually independent of the chosen point in the orbit: if $y \in X$ is in the same orbit as x , then $\gamma_y x = y$ for some $\gamma_y \in \Gamma$. The stabilizer subgroup of y is related to that of x via the conjugation $\Gamma_y \gamma_y = \gamma_y \Gamma_x$, and so the homogeneous spaces Γ/Γ_y and Γ/Γ_x are isometric (with the metric induced by d_S) via right multiplication by γ_y .

The reverse inclusion of (2.2) also holds with natural numbers¹⁷:

$$(2.10) \quad \text{Pen}(S, r + s) = \text{Pen}(\text{Pen}(S, r), s)$$

for $S \subset M$ and $r, s \in \mathbb{N}$; more precisely,

$$(2.11) \quad \text{Pen}(S, r + s) \setminus S = \text{Pen}(\text{Pen}(S, r) \setminus S, s) \setminus S.$$

¹⁷On complete path metric spaces, these equality holds for all $r, s \geq 0$.

Note that (2.10) follows from (2.11) and (2.2). The inclusion “ \supset ” of (2.11) is given by (2.2). To prove the reverse inclusion, assume that $r, s > 0$ (if one of $r = 0$ or $s = 0$ there is nothing to prove). If $x \in \text{Pen}(S, r+s) \setminus S$, then there is a finite sequence $z_0, z_1, \dots, z_k = x$ in M with $z_0 \in S$, $k \leq r+s$, and $d(z_{l-1}, z_l) = 1$ for all $l \in \{1, \dots, k\}$, and furthermore that $z_l \in M \setminus S$ for $l \geq 1$. If $k \leq r$, then $x \in \text{Pen}(S, r) \setminus S \subset \text{Pen}(\text{Pen}(S, r) \setminus S, s) \setminus S$; and if $k > r$, then $z_r \in \text{Pen}(S, r)$ and $d(x, z_r) \leq k - r \leq s$. This implies that $x \in \text{Pen}(\text{Pen}(S, r) \setminus S, s) \setminus S$, which concludes the proof of (2.11).

LEMMA 2.48. *Let A be a K -net in M for some $K \in \mathbb{N}$. Then, for every $S \subset M$ and all natural $r \geq K$, the set $\text{Pen}(S, r) \cap A$ is a $2K$ -net in $\text{Pen}(S, r)$; in particular, for every $x \in M$, $\overline{B}(x, r) \cap A$ is a $2K$ -net in $\overline{B}(x, r)$.*

PROOF. For each $x \in \text{Pen}(S, r)$ there is $y \in \text{Pen}(S, r-K)$ with $d(x, y) \leq K$ by (2.10), and there is $z \in A$ with $d(y, z) \leq K$ since A is a K -net. So $z \in \text{Pen}(S, r) \cap A$ by (2.10), and $d(x, z) \leq 2K$ by the triangle inequality. \square

Suppose from now on that G is of *finite type* in the sense that there is some $K \in \mathbb{N}$ such that each vertex meets at most K edges, and assume that $K \geq 2$ (if $K = 0$, then G has just one vertex and no edges; if $K = 1$, then G has at most two vertices and one edge joining them). For each $r \in \mathbb{N}$, let

$$(2.12) \quad \Lambda_{K,r} = \begin{cases} 1 + K \frac{(K-1)^r - 1}{K-2} & \text{if } K > 2 \\ 1 + 2r & \text{if } K = 2. \end{cases}$$

Then

$$(2.13) \quad |\overline{B}(x, r)| \leq 1 + K + K(K-1) + \dots + K(K-1)^{r-1} = \Lambda_{K,r}$$

for all $x \in M$ and $r \in \mathbb{N}$. Therefore

$$(2.14) \quad |\text{Pen}(S, r)| \leq \Lambda_{K,r} |S|$$

for any $S \subset M$ and $r \in \mathbb{N}$.

The growth type of the function $r \mapsto |B(x, r)|$ ($r \geq 1$) is independent of the choice of $x \in M$, and is called the *growth type* of M (as set of vertices of a connected graph¹⁸), or of G (as graph).

The *boundary* of a subset $S \subset M$ is $\partial S = \partial_1 S$. The sets $\partial S \cap S$ and $\partial S \setminus S$ are respectively called *inner* and *outer boundaries*. Since

$$\partial S \cap S \subset \text{Pen}(\partial S \setminus S, 1), \text{ and } \partial S \setminus S \subset \text{Pen}(\partial S \cap S, 1),$$

it follows by (2.14) that

$$(2.15) \quad \frac{1}{\Lambda_{K,1}} |\partial S \setminus S| \leq |\partial S \cap S| \leq \Lambda_{K,1} |\partial S \setminus S|.$$

LEMMA 2.49. $\partial_r S = \text{Pen}(\partial S, r-1)$ for all $r \in \mathbb{Z}^+$.

¹⁸This definition is indeed valid for any metric space with finite balls.

PROOF. By (2.11),

$$\begin{aligned}
\partial_r S &= \text{Pen}(S, r) \cap \text{Pen}(M \setminus S, r) \\
&= ((\text{Pen}(S, r) \cap \text{Pen}(M \setminus S, r)) \setminus S) \\
&\quad \cup (\text{Pen}(S, r) \cap \text{Pen}(M \setminus S, r) \cap S) \\
&= (\text{Pen}(S, r) \setminus S) \cup (\text{Pen}(M \setminus S, r) \cap S) \\
&= (\text{Pen}(\text{Pen}(S, 1) \setminus S, r-1) \setminus S) \\
&\quad \cup (\text{Pen}(\text{Pen}(M \setminus S, 1) \cap S, r-1) \cap S) \\
&= (\text{Pen}(\partial S \setminus S, r-1) \setminus S) \cup (\text{Pen}(\partial S \cap S, r-1) \cap S) \\
&\subset \text{Pen}(\partial S, r-1).
\end{aligned}$$

On the other hand, by (2.1) and (2.10),

$$\begin{aligned}
\text{Pen}(\partial S, r-1) &= \text{Pen}(\text{Pen}(S, 1) \cap \text{Pen}(M \setminus S, 1), r-1) \\
&\subset \text{Pen}(\text{Pen}(S, 1), r-1) \cap \text{Pen}(\text{Pen}(M \setminus S, 1), r-1) \\
&= \text{Pen}(S, r) \cap \text{Pen}(M \setminus S, r) = \partial_r S. \quad \square
\end{aligned}$$

Lemma 2.49 and (2.14) give

$$(2.16) \quad |\partial_r S| \leq \Lambda_{K, r-1} |\partial S|.$$

The metric space M (as set of vertices of a connected graph) is called *Følner* if it contains a sequence of finite subsets S_n such that $|\partial S_n|/|S_n| \rightarrow 0$ as $n \rightarrow \infty$.

Let $\{G_i\}$ be a class of connected graphs, and let $\{M_i\}$ be the class of metric spaces defined by their vertices. Then $\{G_i\}$ is said to be of *equi-finite type* if there is $K \in \mathbb{N}$ such that each vertex at each G_i meets at most K edges. The class $\{M_i\}$ is called *equi-Følner* if each M_i has a Følner sequence $S_{i,n}$ such that, for some $a > 0$, $|\partial S_{i,n}|/|S_{i,n}| \leq a |\partial S_{j,n}|/|S_{j,n}|$ for all i and j .

2.4.2. Metric spaces of coarse bounded geometry.

DEFINITION 2.50 (Block-Weinberger [BW92]). A *quasi-lattice* Γ of M is an R -net of M for some $R \geq 0$ such that $|\Gamma \cap \overline{B}(x, r)| \leq Q_r$ for every $x \in M$, where $r \mapsto Q_r$ ($r, Q_r \geq 0$) is a mapping independent of x ; the term (R, Q_r) -*quasi-lattice* may be also used. It is said that M is of *coarse bounded geometry* if it has an (R, Q_r) -quasi-lattice for some (R, Q_r) ; in this case, (R, Q_r) is called a *coarse bound* of M .

EXAMPLE 2.51 (Block-Weinberger [BW92]). (i) If M is the metric space of vertices of any connected graph G , then M is of coarse bounded geometry if and only if G is of finite type; indeed, if each vertex meets at most K edges, then M is a $(0, \Lambda_{K,r})$ -quasi-lattice in itself by (2.13), where $\Lambda_{K,r}$ is given by (2.12).

(ii) If M is a connected complete Riemannian manifold with a positive injectivity radius and whose Ricci curvature is bounded from below, then it is of coarse bounded geometry; in particular, this holds if M is of bounded geometry¹⁹.

¹⁹Recall that a Riemannian manifold is said to be of *bounded geometry* if it has a positive injectivity radius and each covariant derivative of arbitrary order of its curvature tensor is uniformly bounded.

DEFINITION 2.52. The class $\{M_i\}$ is said to be of *equi-coarse bounded geometry* when the metric spaces M_i are of coarse bounded geometry with a common coarse bound (R, Q_r) . In this case, a class $\{\Gamma_i \subset M_i\}$ of (R, Q_r) -quasi-lattices is called a class of *equi-quasi-lattices*.

EXAMPLE 2.53. If $\{M_i\}$ is the class of metric spaces of vertices of a class of corresponding connected graphs $\{G_i\}$, then $\{M_i\}$ is of equi-coarse bounded geometry if and only if $\{G_i\}$ is of equi-finite type. If a class of connected complete Riemannian manifolds have injectivity radius bounded from below by a common positive constant, and Ricci curvature uniformly bounded from below by a common constant, then they are of equi-coarse bounded geometry.

PROPOSITION 2.54. Suppose that Γ is an (R, Q_r) -quasi-lattice of M , $f : M \rightarrow M'$ is an (s_r, t_r) -rough map, and that there is a rough map $g : M' \rightarrow M$ such that gf and fg are c -close to id_M and $\text{id}_{M'}$, respectively. Then $\Gamma' = f(\Gamma)$ is an (R', Q'_r) -quasi-lattice of M' , with $R' = c + s_R$ and $Q'_r = Q_{t_r+R'}$.

PROOF. For each $x' \in M'$, there is $y \in \Gamma$ such that $d(g(x'), y) \leq R$. Then $y' = f(y) \in \Gamma'$ and

$$d'(x', y') \leq d'(x', fg(x')) + d'(fg(x'), y') \leq c + s_R.$$

Moreover

$$\Gamma' \cap \overline{B}_{M'}(x', r) \subset \overline{B}_{\Gamma'}(y', r + R') \subset f(\overline{B}_\Gamma(y, t_r + R'))$$

for all $r \geq 0$, obtaining

$$|\Gamma' \cap \overline{B}_{M'}(x', r)| \leq |\Gamma \cap \overline{B}_M(y, t_r + R')| \leq Q_{t_r+R'} . \quad \square$$

COROLLARY 2.55. If Γ is an (R, Q_r) -quasi-lattice of M , and $\phi : M \rightarrow M'$ is a (λ, b, c) -large scale Lipschitz equivalence, then $\Gamma' = \phi(\Gamma)$ is an (R', Q'_r) -quasi-lattice of M' , where $R' = \lambda R + b + c$ and $Q'_r = Q_{\lambda(r+R') + b + 2c}$.

PROOF. By Proposition 2.39, we can apply Proposition 2.54 with $s_r = \lambda r + b$ and $t_r = \lambda r + b + 2c$. \square

COROLLARY 2.56. If M is of coarse bounded geometry with coarse bound (R, Q_r) , and there is a (K, C) -coarse quasi-isometry of M to M' , then M' is of coarse bounded geometry with coarse bound (R', Q'_r) , where

$$R' = K + CR + 2CK, \quad Q'_r = Q_{C(r+R') + 2CK + 2K}.$$

PROOF. By Proposition 2.26, there is a $(C, 2CK, K)$ -large scale Lipschitz equivalence $\phi : M \rightarrow M'$. Let Γ be an (R, Q_r) -quasi-lattice of M . Then $\phi(\Gamma)$ is an (R', Q'_r) -quasi-lattice of M' by Corollary 2.55 \square

REMARK 2.57. According to Corollary 2.56, equi-coarse bounded geometry is preserved by equi-coarse quasi-isometries.

REMARK 2.58. Another version of Corollary 2.56, with $R' = 6C \max\{R, K\}$ and $Q'_r = Q_{C(r+R')}$, can be proved without passing to large scale equivalences.

2.4.3. Coarsely quasi-symmetric metric spaces.

DEFINITION 2.59. Let \mathcal{T} be a set of coarsely quasi-isometric transformations of M , let Φ be a set of maps $M \rightarrow M$, and let $R \geq 0$. The set \mathcal{T} is called:

- *transitive* when, for all $x, y \in M$, there is some $f \in \mathcal{T}$ such that $x \in \text{dom } f$, $y \in \text{im } f$ and $f(x) = y$; and
- *R -quasi-transitive* if, for all $x, y \in M$, there is some $f \in \mathcal{T}$ and $z \in \text{dom } f$ such that $d(x, z) \leq R$ and $d(f(z), y) \leq R$.

The set Φ is called:

- *transitive* when, for all $x, y \in M$, there is some $\phi \in \Phi$ with $\phi(x) = y$; and
- *R -quasi-transitive* if, for all $x, y \in M$, some $\phi \in \Phi$ satisfies $d(f(x), y) \leq R$.

DEFINITION 2.60. A metric space M is called *coarsely quasi-symmetric* if there is a transitive set of equi-coarsely quasi-isometric transformations of M .

DEFINITION 2.61. A class $\{M_i\}$ is called *equi-coarsely quasi-symmetric* if, for some $K \geq 0$ and $C \geq 1$, there is a transitive class of (K, C) -coarsely quasi-isometric transformations of every M_i .

LEMMA 2.62. *The following statements are equivalent:*

- (i) $\{M_i\}$ is equi-coarsely quasi-symmetric.
- (ii) For some $R, K \geq 0$ and $C \geq 1$, there is an R -quasi-transitive class of (K, C) -quasi-isometric transformations of each M_i .
- (iii) For some $\lambda \geq 1$ and $b, c \geq 0$, there is a transitive class of (λ, b, c) -large scale Lipschitz transformations of each M_i .
- (iv) For some $R, b, c \geq 0$ and $\lambda \geq 1$, there is an R -quasi-transitive class of (λ, b, c) -large scale Lipschitz transformations of each M_i .

PROOF. This follows from Propositions 2.26, 2.27 and 2.31, and Corollary 2.33. \square

PROPOSITION 2.63. *(Equi-)coarse quasi-symmetry is preserved by (equi-)coarse quasi-isometries.*

PROOF. Assume that there is some transitive set Φ of (λ, b, c) -large scale Lipschitz transformations of M , and there is a (λ, b, c) -large scale Lipschitz equivalence $\xi : M \rightarrow M'$. Let $\zeta : M' \rightarrow M$ be a (λ, b) -large scale Lipschitz map so that $\zeta\xi$ and $\xi\zeta$ are c -close to id_M and $\text{id}_{M'}$, respectively. By Lemma 2.24, it follows that $\Phi' := \{\xi\phi\zeta \mid \phi \in \Phi\}$ is a family of (λ', b', c') -large scale Lipschitz transformations of M' , where (λ', b', c') depends only on (λ, b, c) . For all $x', y' \in M'$, there is some $\phi \in \Phi$ such that $\phi\zeta(x') = \zeta(y')$. Thus $\phi' := \xi\phi\zeta \in \Phi'$ satisfies

$$\begin{aligned} d'(\phi'(x'), y') &\leq d'(\phi'(x'), \xi\zeta(y')) + d'(\xi\zeta(y'), y') \\ &\leq \lambda d(\phi\zeta(x'), \zeta(y')) + b + c = b + c, \end{aligned}$$

obtaining that Φ' is $(b+c)$ -quasi-transitive. Hence the result follows from Lemma 2.62 and Proposition 2.26. \square

REMARK 2.64. A more involved proof can be given by using coarse composites of coarse quasi-isometries, whose coarse distortion is controlled by Proposition 2.12.

2.4.4. Coarsely quasi-convex metric spaces.

DEFINITION 2.65. A metric space, M , is said to be *coarsely quasi-convex* if there are $a, b, c \geq 0$ such that, for each $x, y \in M$, there exists a finite sequence of points $x = x_0, \dots, x_n = y$ in M such that $d(x_{k-1}, x_k) \leq c$ for all $k \in \{1, \dots, n\}$, and

$$\sum_{k=1}^n d(x_{k-1}, x_k) \leq a d(x, y) + b.$$

A class of metric spaces is said to be *equi-coarsely quasi-convex* if all of them satisfy this condition with the same constants a , b , and c .

REMARK 2.66. Coarse quasi-convexity is a coarsely quasi-isometric version of the following condition introduced by Gromov: For each $x, y \in M$ and $\varepsilon > 0$, there is some $z \in M$ such that

$$\max\{d(x, z), d(y, z)\} < \frac{1}{2} d(x, y) + \varepsilon.$$

This property may be called *approximate convexity* because a subset of \mathbf{R}^n satisfies it precisely when said subset has a convex closure. Gromov has shown that a complete metric space is a path metric space if and only if it is approximately convex [Gro99, Theorem 1.8].

REMARK 2.67. The property of coarse quasi-convexity is slightly weaker than the property of monogenicity for coarse spaces defined by metrics [Roe03] (monogenicity means that the condition of Definition 2.65 is satisfied with $a = 1$ and $b = 0$).

EXAMPLE 2.68. Any class of metric spaces, each being the space of vertices of a connected graph, is equi-coarsely quasi-convex (they satisfy the condition of Definition 2.65 with $a = 1$, $b = 0$ and $c = 1$). Of course, any class of connected complete Riemannian manifolds is also coarsely quasi-convex since they are path metric spaces.

PROPOSITION 2.69 (Álvarez-Candel [ALC11, Theorem 3.11]). *A metric space, M , is coarsely quasi-convex if and only if there exists a coarse quasi-isometry of M to the metric space of vertices of some connected graph. A class, $\{M_i\}$, is equi-coarsely quasi-convex if and only if $\{M_i\}$ is equi-coarsely quasi-isometric to a family of metric spaces of vertices of connected graphs.*

REMARK 2.70. In [ALC11, Theorem 3.11], the result was stated using complete path metric spaces instead of graphs, but indeed a graph is constructed in its proof.

REMARK 2.71. Proposition 2.69 is a coarsely quasi-isometric version of [Roe03, Proposition 2.57], which asserts that the monogenic coarse structures are those that are coarsely equivalent to path metric spaces.

REMARK 2.72. As a consequence of Proposition 2.69, we get that (equi-) coarse quasi-convexity is invariant by (equi-) coarse quasi-isometries.

PROPOSITION 2.73 (Álvarez-Candel [ALC11, Proposition 3.15]). *(Equi-)uniformly expansive maps with (equi-)coarsely convex domains are (equi-)large scale Lipschitz.*

COROLLARY 2.74 (Álvarez-Candel [ALC11, Corollary 3.16]). *(Equi-)coarse equivalences with (equi-)coarsely convex domains are (equi-)large scale Lipschitz equivalences.*

2.5. Growth of metric spaces

2.5.1. Growth. Suppose that M and M' are of coarse bounded geometry, and $\{M_i\}$ and $\{M'_i\}$ are of equi-coarse bounded geometry.

For a quasi-lattice Γ of M and $x \in \Gamma$, the function $r \mapsto v_\Gamma(x, r) = |B_\Gamma(x, r)|$ ($r \geq 1$) is called the *growth function* of M induced by Γ and x .

PROPOSITION 2.75. For $k \in \{1, 2\}$, let Γ_k be an (R_k, Q_r^k) -quasi-lattice of M , and $x_k \in \Gamma_k$. Take any $\delta \geq d(x_1, x_2)$. Then, for all $r \geq 1$,

$$v_{\Gamma_1}(x_1, r) \leq Q_{R_2}^1 v_{\Gamma_2}(x_2, r + \delta + R_2).$$

PROOF. Since $B_M(x_1, r) \subset B_M(x_2, r + \delta)$ for all $r \geq 1$, and Γ_2 is an R_2 -net, then

$$B_{\Gamma_1}(x_1, r) \subset \bigcup_{y \in B_{\Gamma_2}(x_2, r + \delta + R_2)} \overline{B}_M(y, R_2) \cap \Gamma_1,$$

which implies the stated inequality. \square

The following definitions are justified by Proposition 2.75.

DEFINITION 2.76. The *growth type* of M is the growth type of $r \mapsto v_\Gamma(x, r)$ for any quasi-lattice Γ of M and $x \in \Gamma$. We may also say that M and M' have *equivalent growth* when they have the same growth type.

DEFINITION 2.77. Two classes of metric spaces, $\{M_i\}$ and $\{M'_i\}$, have *equi-equivalent growth* when there are equi-quasi-lattices, $\Gamma_i \subset M_i$ and $\Gamma'_i \subset M'_i$, and there are points, $x_i \in M_i$ and $x'_i \in M'_i$, such that $r \mapsto v_{\Gamma_i}(x_i, r)$ and $r \mapsto v_{\Gamma'_i}(x'_i, r)$ have equi-equivalent growth.

REMARK 2.78. (i) According to Section 2.1 and Definition 2.76, the following notions make sense for the growth type of M : polynomial of exact degree $d \in \mathbb{N}$, polynomial, exponential, pseudo-quasi-polynomial, subexponential or quasi-polynomial, quasi-exponential and non-exponential.

(ii) For any quasi-lattice Γ of M and $x \in M$, the quantities

$$\limsup_{r \rightarrow \infty} \frac{\log v_\Gamma(x, r)}{\log r}, \quad \liminf_{r \rightarrow \infty} \frac{\log v_\Gamma(x, r)}{\log r}$$

depend only on the growth type of M by (2.6) and (2.7).

(iii) With the notation of Proposition 2.75,

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log v_{\Gamma_1}(x_1, r)}{r} &\leq b \liminf_{r \rightarrow \infty} \frac{\log v_{\Gamma_2}(x_2, r)}{r}, \\ \limsup_{r \rightarrow \infty} \frac{\log v_{\Gamma_1}(x_1, r)}{r} &\leq b \limsup_{r \rightarrow \infty} \frac{\log v_{\Gamma_2}(x_2, r)}{r}, \end{aligned}$$

for any $b > Q_{R_2}^1$ by (2.5), (2.8) and (2.9).

EXAMPLE 2.79. If M is the metric space of vertices of a connected graph G of finite type, then M is a quasi-lattice in itself, and therefore its growth type as metric space of coarse bounded geometry equals its growth type as metric space of vertices of a connected graph. If M is a connected complete Riemannian manifold of bounded geometry, then its growth type as metric space of coarse bounded geometry equals its growth type as Riemannian manifold²⁰.

²⁰Recall that the *growth type* of M , as Riemannian manifold, is the growth type of $r \mapsto \text{vol } B(x, r)$ for any $x \in M$.

PROPOSITION 2.80. *Let Γ be a quasi-lattice of M , $\phi : M \rightarrow M'$ a (λ, b, c) -large scale Lipschitz equivalence, and $x \in \Gamma$. Let $\Gamma' = \phi(\Gamma)$ and $x' = \phi(x)$. Then, for all $r \geq 1$,*

$$v_{\Gamma'}(x', r) \leq v_{\Gamma}(x, \lambda r + b + 2c) .$$

PROOF. The result follows since

$$B_{\Gamma'}(x', r) \subset \phi(B_{\Gamma}(x, \lambda r + b + 2c))$$

for all $r \geq 1$ because ϕ is $(\lambda, b + 2c)$ -large scale bi-Lipschitz by Lemma 2.21. \square

REMARK 2.81. In Proposition 2.80, Γ' is a quasi-lattice of M' by Corollary 2.55.

COROLLARY 2.82. *Let Γ and Γ' be (R, Q_r) -quasi-lattices of M and M' , respectively, and let $\phi : M \rightarrow M'$ be a (λ, b, c) -large scale Lipschitz equivalence. Take $x \in \Gamma$, $x' \in \Gamma'$ and $\delta \geq d'(\phi(x), x')$. Then*

$$v_{\Gamma'}(x', r) \leq p v_{\Gamma}(x, \lambda r + q)$$

for all $r \geq 1$, where

$$p = Q_{\lambda R + b + c} , \quad q = \lambda(\delta + \lambda R + b + c) + b + 2c .$$

PROOF. This is a direct consequence of Corollary 2.55 and Propositions 2.80 and 2.75. \square

COROLLARY 2.83. *Let Γ and Γ' be (R, Q_r) -quasi-lattices of M and M' , respectively, and let $f : A \rightarrow A'$ be a (K, C) -coarse quasi-isometry of M to M' . Take $x \in \Gamma$, $x' \in \Gamma'$, $y \in A$, $\delta \geq d(x, y)$ and $\delta' \geq d'(x', f(y))$. Then*

$$v_{\Gamma'}(x', r) \leq p v_{\Gamma}(x, Cr + q)$$

for all $r \geq 1$, where

$$p = Q_{CR + 2CK + K} , \quad q = C(C\delta + 4CK + 2K + \delta' + CR) + 2CK + 2K .$$

PROOF. By Proposition 2.26, f is induced by a $(C, 2CK, K)$ -large scale Lipschitz equivalence $\phi : M \rightarrow M'$. Moreover

$$\begin{aligned} d'(\phi(x), x') &\leq d'(\phi(x), \phi(y)) + d'(\phi(y), x') \\ &\leq C d(x, y) + 2CK + \delta' \leq C\delta + 2CK + \delta' . \end{aligned}$$

Then the result follows from Corollary 2.82. \square

REMARK 2.84. According to Corollary 2.83, (equi-) coarsely quasi-isometric metric spaces of (equi-) coarse bounded geometry have (equi-) equivalent growth.

REMARK 2.85. The following version of Corollary 2.83 can be proved without using large scale Lipschitz equivalences, but using instead coarse composites (Proposition 2.12): With the hypothesis of Corollary 2.83, if $d(x, y) \leq 2K^*$ and $d'(x', f(y)) \leq 2K^*(5C + 1)$, where $K^* = \max\{R, K\}$, then

$$v_{\Gamma}(x_1, r) \leq Q_{\widehat{K}} \cdot v_{\Gamma'}(x'_2, 250 C^4 r),$$

for all $r \geq \widehat{K}$, where

$$\widehat{K} = 25 K^* (5C + 1) C^2 + 5K^* C .$$

2.5.2. Growth symmetry.

DEFINITION 2.86. A metric space, M , is called *growth symmetric* if there is a quasi-lattice Γ in M so that the growth functions $r \mapsto v_\Gamma(x, r)$, for all $x \in \Gamma$, equi-dominate each other.

REMARK 2.87. (i) Definition 2.86 is independent of the choice of Γ by Proposition 2.75.

(ii) From Proposition 2.75, it also follows that, given quasi-lattices Γ_1 and Γ_2 of M , if all growth functions $r \mapsto v_{\Gamma_1}(x, r)$, with $x \in \Gamma_1$, equi-dominate all growth functions $r \mapsto v_{\Gamma_2}(y, r)$, with $y \in \Gamma_2$, then M is growth-symmetric.

DEFINITION 2.88. A class $\{M_i\}$ is called *equi-growth symmetric* if there are equi-quasi-lattices $\Gamma_i \subset M_i$ so that the growth functions $r \mapsto v_{\Gamma_i}(x, r)$, for all i and $x \in \Gamma_i$, equi-dominate one another.

PROPOSITION 2.89. (*Equi*)-large scale Lipschitz equivalences preserve (*equi*)-growth symmetry.

PROOF. Let Γ and Γ' be (R, Q_r) -quasi-lattices of M and M' , respectively. Assume that there are $a, b, c \geq 1$ such that $v_\Gamma(x, r) \leq a v_{\Gamma'}(y, br)$ for all $r \geq c$ and $x, y \in \Gamma$. Let $\phi : M \rightarrow M'$ and $\psi : M' \rightarrow M$ be (λ, b, c) -large scale Lipschitz equivalences so that $\psi\phi$ and $\phi\psi$ are c -close to id_M and $\text{id}_{M'}$, respectively. For all $x', y' \in \Gamma'$, there are $x, y \in \Gamma$ such that $d'(\phi(x), x') \leq R$ and $d(y, \psi(y')) \leq R$. Let p and q be the constants defined in Corollary 2.82 with $\delta = R$. Applying Corollary 2.82 to ϕ and ψ , it follows that

$$v_{\Gamma'}(x', r) \leq p v_\Gamma(x, \lambda r + q) \leq ap v_{\Gamma'}(y, b(\lambda r + q)) \leq ap^2 v_{\Gamma'}(y', \lambda b(\lambda r + q) + q)$$

if $r \geq \max\{1, \frac{c-q}{\lambda}\}$. \square

COROLLARY 2.90. (*Equi*)-coarse quasi-isometries preserve (*equi*)-growth symmetry.

PROOF. This follows from Propositions 2.89 and 2.26. \square

2.6. Amenable metric spaces

2.6.1. Amenability. Suppose that M and M' are of coarse bounded geometry, and $\{M_i\}$ and $\{M'_i\}$ are of equi-coarse bounded geometry.

DEFINITION 2.91 (Block-Weinberger [BW92]). A metric space, M , is called *amenable* if it has a quasi-lattice Γ and a sequence of finite subsets $S_n \subset \Gamma$ such that $|\partial_r^\Gamma S_n|/|S_n| \rightarrow 0$ as $n \rightarrow \infty$ for each $r > 0$. Such a sequence S_n is called a *Følner sequence* in Γ .

EXAMPLE 2.92. If M is the metric space of vertices of some connected graph of finite type, then M is amenable if and only if it is Følner as metric space of vertices of some connected graph (cf. (2.16)). If M is a connected complete Riemannian manifold of bounded geometry, then it is amenable as metric space of coarse bounded geometry if and only if it is Følner as Riemannian manifold²¹.

DEFINITION 2.93. $\{M_i\}$ is called *weakly equi-amenable* when

²¹Recall that a Riemannian manifold is called *Følner* if it has a sequence of smooth compact domains Ω_n such that $\text{vol } \partial\Omega_n / \text{vol } \Omega_n \rightarrow 0$ as $n \rightarrow \infty$.

- there are equi-quasi-lattices $\Gamma_i \subset M_i$;
- there are subsets $S_{i,m,n} \subset \Gamma_i$ ($m, n \in \mathbb{N}$);
- there is $a \geq 1$, and mappings $r \mapsto p_r$ ($r, p_r > 0$) and $t \mapsto q_t$ ($t, q_t \geq 0$); and
- there is a nonempty subset $\mathcal{N}_{i,j,m,n,t} \subset \mathbb{N}$ for every i, j, m, n and t ;

such that:

- for each i and m , the sequence $S_{i,m,n}$ is Følner in Γ_i ; and,
- for each i, m and t , and any Følner sequence X_n of Γ_i with $S_{i,m,n} \subset X_n \subset \text{Pen}_{\Gamma_i}(S_{i,m,n}, t)$, there is some $Y_{i,j,m',n} \subset \Gamma_j$ for all j, n and $m' \in \mathcal{N}_{i,j,m,n,t}$ so that

$$(2.17) \quad S_{j,m',n} \subset Y_{i,j,m',n} \subset \text{Pen}_{\Gamma_j}(S_{j,m',n}, q_t),$$

$$(2.18) \quad \frac{|\partial_r^{\Gamma_j} Y_{i,j,m',n}|}{|Y_{i,j,m',n}|} \leq a \frac{|\partial_{p_r}^{\Gamma_i} X_n|}{|X_n|},$$

for all r .

If moreover the mappings $r \mapsto p_r$ and $t \mapsto q_t$ can be chosen to be affine, then $\{M_i\}$ is called *equi-amenable*.

PROPOSITION 2.94. *The following are true:*

- (i) *(Equi-)rough equivalences preserve (weak equi-)amenability.*
- (ii) *(Equi-)coarse quasi-isometries preserve (equi-)amenability.*

PROOF. Let $f : M \rightarrow M'$ be an (s_r, c) -rough equivalence. By Definition 2.35, there is an s_r -rough map $g : M' \rightarrow M$ such that gf and fg are c -close to id_M and $\text{id}_{M'}$, respectively. Let Γ be an (R, Q_r) -quasi-lattice of M , for some coarse bound (R, Q_r) . By Proposition 2.54, $\Gamma' = f(\Gamma)$ is an (R', Q'_r) -quasi-lattice of M' , where $R' = c + s_R$ and $Q'_r = Q_{s_r+R'}$.

Let $S \subset \Gamma$ be finite and let $S' = f(S) \subset \Gamma'$. Then $S \subset \text{Pen}_{\Gamma}(g(S'), c)$, and so

$$(2.19) \quad |S| \leq Q_c |S'|,$$

CLAIM 1. $\partial_r^{\Gamma'} S' \subset f(\partial_{s_r}^{\Gamma} S)$ for all $r > 0$.

Indeed, let $x' \in \partial_r^{\Gamma'} S'$ and $x \in \Gamma$ with $f(x) = x'$. There are points $y' \in S'$ and $z' \in \Gamma' \setminus S'$ such that $d'(x', y') \leq r$ and $d'(x', z') \leq r$. If $y \in S$ and $z \in \Gamma \setminus S$ are such that $f(y) = y'$ and $f(z) = z'$, then $d(x, y) \leq s_r$ and $d(x, z) \leq s_r$, and so $x \in \partial_{s_r}^{\Gamma} S$.

It follows from Claim 1 that

$$(2.20) \quad |\partial_r^{\Gamma'} S'| \leq |\partial_{s_r}^{\Gamma} S|,$$

and combining (2.19) and (2.20) that

$$(2.21) \quad \frac{|\partial_r^{\Gamma'} S'|}{|S'|} \leq Q_c \frac{|\partial_{s_r}^{\Gamma} S|}{|S|}.$$

If S_n is a Følner sequence in Γ , then S'_n is a Følner sequence in Γ' by (2.21), and therefore M' is amenable. This shows the weaker version of (i). Then the weaker version of (ii) follows by Propositions 2.26 and 2.39-(ii).

CLAIM 2. $f(\partial_r^{\Gamma} \text{Pen}_{\Gamma}(S, 2c)) \subset \partial_{u_r}^{\Gamma'} S'$ for all $r > 0$, where $u_r = s_r + s_{2c}$.

Let $x \in \partial_r^{\Gamma} \text{Pen}_{\Gamma}(S, 2c)$. There are some $y \in \text{Pen}_{\Gamma}(S, 2c)$ and $z \in \Gamma \setminus \text{Pen}_{\Gamma}(S, 2c)$ such that $d(x, y) \leq r$ and $d(x, z) \leq r$. Then there is some $y_0 \in S$ so that $d(y, y_0) \leq 2c$.

Let $x' = f(x)$, $y' = f(y)$ and $z' = f(z)$ in Γ' , and $y'_0 = f(y_0)$ in S' . We have $z' \in \Gamma' \setminus S'$; otherwise, $z' = f(z_0)$ for some $z_0 \in S$, obtaining

$$d(z, z_0) \leq d(z, g(z')) + d(g(z'), z_0) \leq 2c,$$

which is a contradiction because $z \notin \text{Pen}_\Gamma(S, 2c)$. Moreover

$$d'(x', y'_0) \leq d'(x', y') + d'(y', y'_0) \leq s_r + s_{2c} = u_r$$

and $d'(x', z') \leq s_r$, giving $x' \in \partial_{u_r}^{\Gamma'} S'$, which shows Claim 2.

Applying (2.19) to the set $\partial_r^\Gamma \text{Pen}_\Gamma(S, 2c)$, and using Claim 2, we get

$$(2.22) \quad \frac{|\partial_r^\Gamma \text{Pen}_\Gamma(S, 2c)|}{|\text{Pen}_\Gamma(S, 2c)|} \leq Q_c \frac{|f(\partial_r^\Gamma \text{Pen}_\Gamma(S, 2c))|}{|S|} \leq Q_c \frac{|\partial_{u_r}^{\Gamma'} S'|}{|S'|}.$$

Suppose that $\{M_i\}$ satisfies the condition of weak equi-amenability (Definition 2.93) with a class $\{\Gamma_i\}$ of corresponding (R, Q_r) -quasi-lattices, subsets $S_{i,m,n} \subset \Gamma_i$, a constant $a \geq 1$, mappings $r \mapsto p_r$ and $t \mapsto q_t$, and nonempty subsets $\mathcal{N}_{i,j,m,n,t} \subset \mathbb{N}$. Consider a family of equi-rough equivalences $f_i : M_i \rightarrow M'_i$ with common rough equivalence distortion (s_r, c) . By Proposition 2.54, each $\Gamma'_i = f_i(\Gamma_i)$ is an (R', Q'_r) -quasi-lattice of M'_i , where $R' = c + s_R$ and $Q'_r = Q_{s_r+R'}$. By (2.21), for each i and m , $S'_{i,m,n} = f_i(S_{i,m,n})$ is a Følner sequence in Γ'_i . Given i, m and $t' \geq 0$, take any Følner sequence X'_n in any Γ'_i with $S'_{i,m,n} \subset X'_n \subset \text{Pen}_{\Gamma'_i}(S'_{i,m,n}, t')$. Then

$$X_n := \text{Pen}_{\Gamma_i}(f_i^{-1}(X'_n), 2c) \subset \text{Pen}_{\Gamma_i}(S_{i,m,n}, s_{t'} + 2c);$$

in particular, every X_n is finite. Moreover X_n is a Følner sequence in Γ_i by (2.22) since $f_i(f_i^{-1}(X'_n)) = X'_n$. Thus (2.17) and (2.18) are satisfied with $t = s_{t'} + 2c$ and some subsets $Y_{j,m',n} \subset \Gamma_j$ ($m' \in \mathcal{N}_{i,j,m,n}$). Let $q_{t'} = s_{q_{t'}}$ and $Y'_{j,m',n} = f_j(Y_{j,m',n}) \subset \Gamma'_j$ for $m' \in \mathcal{N}_{i,j,m,n}$. We have

$$S'_{j,m',n} \subset Y'_{j,m',n} \subset \text{Pen}_{\Gamma'_j}(S'_{j,m',n}, q_{t'}).$$

For $r' > 0$, let $r = s_{r'}$ and $p_{r'} = u_{p_{r'}}$. By (2.21), (2.18) and (2.22),

$$\frac{|\partial_{r'}^{\Gamma'_j} Y'_{j,m',n}|}{|Y'_{j,m',n}|} \leq Q_c \frac{|\partial_r^{\Gamma_j} Y_{j,m',n}|}{|Y_{j,m',n}|} \leq aQ_c \frac{|\partial_{p_r}^{\Gamma_i} X_n|}{|X_n|} \leq aQ_c^2 \frac{|\partial_{p_{r'}}^{\Gamma_i} X'_n|}{|X'_n|}.$$

for all r' . So $\{M'_i\}$ is weakly equi-amenable (the stronger version of (i)).

The stronger version of (ii) follows like the stronger version of (i), assuming that the family $\{f_i\}$ is equi-large scale Lipschitz by Propositions 2.26 and 2.27, and using the expression of s_r in Proposition 2.39-(ii) and the expression of u_r in Claim 2. \square

REMARK 2.95. In the proof of Proposition 2.94, since the fibers of f are of diameter $\leq s_0$, we could use Q_{s_0} instead of Q_c in (2.19), and Claim 2 could be stated with s_0 instead of $2c$.

REMARK 2.96. The weaker version of Proposition 2.94-(i) was proved by Block and Weinberger [BW92]. Their proof has the following three steps. First, they introduce the uniformly finite homology, $H_\bullet^{\text{uf}}(M)$; second, they show that $H_\bullet^{\text{uf}}(M) \cong H_\bullet^{\text{uf}}(M')$ if M and M' are roughly equivalent; and third, they prove that M is amenable if and only if $H_0^{\text{uf}}(M) \neq 0$.

PROPOSITION 2.97. *If M is amenable, then each quasi-lattice of M has a Følner sequence.*

PROOF. Let Γ be a quasi-lattice of M . Since M is amenable, Γ is also amenable by Proposition 2.94. Therefore some (R, Q_r) -quasi-lattice Γ_0 of Γ has a Følner sequence $S_{0,n}$. It has to be shown that Γ also has a Følner sequence. Let $S_n = \text{Pen}_\Gamma(S_{0,n}, R)$.

CLAIM 3. $\partial_r^\Gamma S_n \subset \text{Pen}_\Gamma(\partial_{r+2R}^{\Gamma_0} S_{0,n}, R)$ for all $r > 0$.

For each $x \in \partial_r^\Gamma S_n$, there are $x_0 \in \Gamma_0$, $y \in S_n$ and $z \in \Gamma \setminus S_n$ such that $d(x, x_0) \leq R$, $d(x, y) \leq r$ and $d(x, z) \leq r$. Then there are points $y_0 \in S_{0,n}$ and $z_0 \in \Gamma_0$ such that $d(y, y_0) \leq R$ and $d(z, z_0) \leq R$. Observe that $z_0 \notin S_{0,n}$ because $z \notin S_n$ and $d(z, z_0) \leq R$. By the triangle inequality, the distances $d(x_0, y_0) \leq r + 2R$ and $d(x_0, z_0) \leq r + 2R$, and so $x_0 \in \partial_{r+2R}^{\Gamma_0} S_{0,n}$. Hence $x \in \text{Pen}_\Gamma(\partial_{r+2R}^{\Gamma_0} S_{0,n}, R)$, which completes the proof of Claim 3.

By Claim 3,

$$(2.23) \quad \frac{|\partial_r^\Gamma S_n|}{|S_n|} \leq \frac{|\text{Pen}_\Gamma(\partial_{r+2R}^{\Gamma_0} S_{0,n}, R)|}{|S_{0,n}|} \leq Q_R \frac{|\partial_{r+2R}^{\Gamma_0} S_{0,n}|}{|S_{0,n}|}.$$

Then S_n is a Følner sequence in Γ because $S_{0,n}$ is a Følner sequence in Γ_0 . \square

REMARK 2.98. In the proof of Proposition 2.97, it is easy to check that $\partial_r^{\Gamma_0}(S_n \cap \Gamma_0) \subset \partial_r^\Gamma S_n$ for all $r > 0$, and therefore

$$\frac{|\partial_r^{\Gamma_0}(S_n \cap \Gamma_0)|}{|S_n \cap \Gamma_0|} \leq \frac{|\partial_r^\Gamma S_n|}{|S_n|}.$$

2.6.2. Amenable symmetry.

DEFINITION 2.99. A metric space, M , is called *weakly amenably symmetric* if there are:

- a quasi-lattice Γ of M ;
- subsets $S_{m,n} \subset \Gamma$ ($m, n \in \mathbb{N}$);
- some $a \geq 1$, and mappings $r \mapsto p_r$ ($r, p_r > 0$) and $t \mapsto q_t$ ($t, q_t \geq 0$); and
- a nonempty subset $\mathcal{N}_{m,n,t} \subset \mathbb{N}$ for each m, n and t ;

such that:

- for each m , the sequence $S_{m,n}$ is Følner in Γ_i ;
- for each m, n and $t \geq 0$, $\bigcup_{m' \in \mathcal{N}_{m,n,t}} S_{m',n}$ is a net in M ; and
- for each m and $t \geq 0$, and any Følner sequence X_n of Γ with $S_{m,n} \subset X_n \subset \text{Pen}_\Gamma(S_{m,n}, t)$, there is some $Y_{m',n} \subset \Gamma$ for each n and $m' \in \mathcal{N}_{m,n}$ so that

$$(2.24) \quad S_{m',n} \subset Y_{m',n} \subset \text{Pen}_{\Gamma_i}(S_{m',n}, q_t),$$

$$(2.25) \quad \frac{|\partial_r^\Gamma Y_{m',n}|}{|Y_{m',n}|} \leq a \frac{|\partial_{p_r}^\Gamma X_n|}{|X_n|},$$

for all r .

In this case, Γ is called *weakly Følner symmetric*. If moreover the mappings $r \mapsto p_r$ and $t \mapsto q_t$ can be chosen to be affine, then M is called *amenable symmetric* and Γ is called *Følner symmetric*.

EXAMPLE 2.100. Let M be the union of a plane and an orthogonal line is a metric subspace in \mathbb{R}^3 . Then M is amenable but not amenably symmetric.

DEFINITION 2.101. A class of metric spaces, $\{M_i\}$, is (weakly) *equi-amenably symmetric* if every M_i satisfies the conditions of (weak) amenable symmetry (Definition 2.99) with equi-quasi-lattices $\Gamma_i \subset M_i$, subsets $S_{i,m,n} \subset \Gamma_i$, the same constant $a \geq 1$, the same mappings $r \mapsto p_r$ and $t \mapsto q_t$, and subsets $\mathcal{N}_{i,m,n,t} \subset \mathbb{N}$, such that, for some $L_n, L_{i,m,n,t} \in \mathbb{N}$, $\bigcup_m S_{i,m,n}$ is an L_n -net of Γ_i for all i , and $\bigcup_{m' \in \mathcal{N}_{i,j,m,n,t}} S_{j,m',n}$ is an $L_{i,m,n,t}$ -net of Γ_j for all j .

REMARK 2.102. In Definition 2.101, every M_i satisfies the conditions of (weak) amenable symmetry with subsets $\mathcal{N}_{i,m,n,t} \subset \mathbb{N}$ that may depend on i .

PROPOSITION 2.103. (i) (Equi-)rough equivalences preserve weak (equi-)amenable symmetry.

(ii) (Equi-)coarse quasi-isometries preserve (equi-)amenable symmetry.

PROOF. Let $f : M \rightarrow M'$ be a rough equivalence with rough equivalence distortion (s_r, c) . Suppose that M satisfies the conditions to be weakly amenable symmetric with a quasi-lattice Γ , sets $S_{m,n}$, a constant $a \geq 1$, mappings $r \mapsto p_r$ and $t \mapsto q_t$, and subsets $\mathcal{N}_{m,n,t} \subset \mathbb{N}$; in particular, each union $\bigcup_{m' \in \mathcal{N}_{m,n,t}} S_{m',n}$ is a $K_{m,n,t}$ -net in M for some $K_{m,n,t} \geq 0$. Let $S'_{m,n} = f(S_{m,n})$. For each m , the sequence $S'_{m,n}$ is Følner in Γ' by (2.21). Moreover $\bigcup_{m' \in \mathcal{N}_{m,n,t}} S'_{m',n}$ is a $(s_{K_{m,n,t}} + c)$ -net in M' , which can be proved as follows. For each $x' \in M'$, there is some $y \in \bigcup_{m' \in \mathcal{N}_{m,n,t}} S_{m',n}$ such that $d(y, g(x')) \leq K_n$. Then $f(y) \in \bigcup_{m' \in \mathcal{N}_{m,n,t}} S'_{m',n}$ and

$$d'(f(y), x') \leq d'(f(y), fg(x')) + d'(fg(x'), x') \leq s_{K_{m,n,t}} + c.$$

The rest of the proof is analogous to the proof of Proposition 2.94. \square

PROPOSITION 2.104. If M is (weakly) amenable symmetric, then every quasi-lattice of M is (weakly) Følner symmetric.

PROOF. Suppose that M is (weakly) amenable symmetric, and let Γ be a quasi-lattice of M . By Proposition 2.103, Γ satisfies the condition of (weak) amenable symmetry (Definition 2.99) with some (R, Q_R) -quasi-lattice Γ' of Γ , a family of Følner sequences $S'_{m,n}$ in Γ' , some constant $a' \geq 1$, and some mappings $r \mapsto p'_r$ and $t \mapsto q'_t$. For each m and n , let $S_{m,n} = \text{Pen}_\Gamma(S'_{m,n}, R)$. For every m , $S_{m,n}$ is a Følner sequence in Γ by (2.23). Given m and $t \geq 0$, let X_n be a Følner sequence in Γ with $S_{m,n} \subset X_n \subset \text{Pen}_\Gamma(S_{m,n}, t)$. Then $X'_n := X_n \cap \Gamma'$ is a Følner sequence in Γ' by Remark 2.98, and moreover $S'_{m,n} \subset X'_n \subset \text{Pen}_{\Gamma'}(S'_{m,n}, t + R)$. Hence X'_n satisfies (2.24) and (2.25) with some subsets $Y'_{m,n} \subset \Gamma'$, using q'_{t+R} . For $Y_{m,n} = \text{Pen}_\Gamma(Y'_{m,n}, R)$, we have

$$S_{m,n} \subset Y_{m,n} \subset \text{Pen}_\Gamma(\text{Pen}_{\Gamma'}(S'_{m,n}, q'_{t+R}), R) \subset \text{Pen}_\Gamma(S_{m,n}, q'_{t+R} + R),$$

$$\frac{|\partial_r^\Gamma Y_{m,n}|}{|Y_{m,n}|} \leq Q_R \frac{|\partial_r^{\Gamma'} Y'_{m,n}|}{|Y'_{m,n}|} \leq a' Q_R \frac{|\partial_{p'_r}^{\Gamma'} X'_n|}{|X'_n|} \leq a' Q_R \frac{|\partial_{p'_r}^\Gamma X_n|}{|X_n|},$$

by (2.2) and (2.23)–(2.25). Thus the condition of (weak) amenable symmetry of M is satisfied with Γ . \square

DEFINITION 2.105. A class of metric spaces, $\{M_i\}$, is *jointly weakly amenable symmetric* if there are:

- equi-quasi-lattices $\Gamma_i \subset M_i$;
- subsets $S_{i,m,n} \subset \Gamma_i$ ($m, n \in \mathbb{N}$);
- mappings $r \mapsto p_r$ ($r, p_r > 0$) and $t \mapsto q_t$ ($t, q_t \geq 0$);

- nonempty subsets $\mathcal{N}_{i,j,m,n,t} \subset \mathbb{N}$, one for each i, j, m, n and t ;
- numbers $L_n, L_{i,m,n,t} \in \mathbb{N}$, one for each i, m, n and t ; and $a \geq 1$,

such that:

- for each i and m , the sequence $S_{i,m,n}$ is Følner in Γ_i ;
- for each i and n , $\bigcup_m S_{i,m',n}$ is an L_n -net in M_i ;
- for each i, j, m, n and t , $\bigcup_{m' \in \mathcal{N}_{i,j,m,n,t}} S_{j,m',n}$ is an $L_{i,m,n,t}$ -net in M_j ; and
- for each i, m and t , and any Følner sequence X_n of Γ_i with $S_{i,m,n} \subset X_n \subset \text{Pen}_{\Gamma_i}(S_{i,m,n}, t)$, there is some $Y_{i,j,m',n} \subset \Gamma_j$ for all j, n and $m' \in \mathcal{N}_{m,n}$ so that

$$S_{j,m',n} \subset Y_{i,j,m',n} \subset \text{Pen}_{\Gamma_j}(S_{i,m',n}, q_t),$$

$$\frac{|\partial_r^\Gamma Y_{i,j,m',n}|}{|Y_{i,j,m',n}|} \leq a \frac{|\partial_{p_r}^\Gamma X_n|}{|X_n|},$$

for all r .

In this case, $\{\Gamma_i\}$ is called *jointly weakly Følner symmetric*. If moreover the mappings $r \mapsto p_r$ and $t \mapsto q_t$ can be chosen to be affine, then $\{M_i\}$ is called *jointly amenably symmetric* and $\{\Gamma_i\}$ *jointly Følner symmetric*.

REMARK 2.106. Joint (weak) amenable symmetry is stronger than (weak) equi-amenability and (weak) equi-amenable symmetry.

PROPOSITION 2.107. (i) *Equi-rough equivalences preserve weak joint amenable symmetry.*

(ii) *Equi-coarse quasi-isometries preserve joint amenable symmetry.*

PROOF. Similar to the proof of Proposition 2.103. \square

2.7. Coarse ends

The end space of a topological space X is constructed as follows. Let \mathcal{K} be the family of compact subsets of X . For each $K \in \mathcal{K}$, let \mathcal{U}_K be the discrete space of connected components of $X \setminus K$ with non-compact closure. If $K \subset L$ in \mathcal{K} , we get a map $\eta_{K,L} : \mathcal{U}_L \rightarrow \mathcal{U}_K$ determined by $\eta_{K,L}(U) \supset U$. These maps form an inverse system, whose inverse limit is the space of *ends* of X , denoted by $\mathcal{E}(X)$, which is Hausdorff and totally disconnected. Thus any $\mathbf{e} \in \mathcal{E}(X)$ can be described as a map defined on \mathcal{K} such that $\mathbf{e}(K) \in \mathcal{U}_K$ and $\mathbf{e}(K) \supset \mathbf{e}(L)$ if $K \subset L$ in \mathcal{K} . Suppose that X has an increasing sequence of compact subsets, (K_n) , whose interiors cover X . Then the topology of $\mathcal{E}(X)$ is induced by the ultrametric $d_{(K_n)}$ defined by

$$d_{(K_n)}(\mathbf{e}, \mathbf{f}) = \exp(-\sup\{n \in \mathbb{N} \mid \mathbf{e}(K_n) = \mathbf{f}(K_n)\}).$$

A coarse version of these concepts for metric spaces is obtained by replacing compact sets by bounded sets, as shown next.

2.7.1. Coarse connectivity.

DEFINITION 2.108. Let $\mu > 0$. Two points $x, y \in M$ are *coarsely μ -connected* if there is, for some k , a finite sequence $\{z_l\}_{l=0}^k$ in M such that $x = z_0, z_1, \dots, z_k = y$ and that $d(z_{l-1}, z_l) \leq \mu$ for all $l \in \{1, \dots, k\}$. This concept defines an equivalence relation on M whose equivalence classes are called *coarse μ -connected components*. If all points in M are μ -connected, then M is called *coarsely μ -connected*. If M is coarsely μ -connected for some $\mu > 0$, then M is called *coarsely connected*.

REMARK 2.109. The following properties are elementary:

- (i) The coarse μ -connected components are the maximal coarse μ -connected subsets.
- (ii) If M is coarsely μ -connected, then it is coarsely ν -connected for all $\nu \geq \mu$.
- (iii) If M is coarsely μ -connected, then, for all non-trivial partition of M into two sets, $\{A, B\}$, there is some $x \in A$ and $y \in B$ such that $d(x, y) \leq \mu$.
- (iv) If A and B are coarsely μ -connected subsets of M , and $d(x, y) \leq \mu$ for some $x \in A$ and $y \in B$, then $A \cup B$ is coarsely μ -connected.

REMARK 2.110. Coarse connectivity of M means that the coarse space $[M]$ is monogenic [Roe03], but we prefer the term coarse connectivity because it plays the same role here as connectivity in the definition of ends.

LEMMA 2.111. *If M is coarsely μ -connected, $f : M \rightarrow M'$ satisfies the condition of uniform expansiveness with the mapping s_r , and $f(M)$ is a c -net in M' , then M' is coarsely μ' -connected, where $\mu' = \max\{s_\mu, c\}$.*

PROOF. For $x', y' \in M'$, there are points $x, y \in M$ such that $d'(x', f(x)) \leq c$ and $d'(y', f(y)) \leq c$, and there is a finite sequence $x = z_0, z_1, \dots, z_k = y$ in M such that $d(z_{l-1}, z_l) \leq \mu$ for all $l \in \{1, \dots, k\}$. Then $d'(f(z_{l-1}), f(z_l)) \leq s_\mu$, obtaining that x' is coarsely μ' -connected to y' . \square

COROLLARY 2.112. *Coarse connectivity is invariant by coarse equivalences.*

REMARK 2.113. Corollary 2.112 is indeed trivial by Remark 2.110.

LEMMA 2.114. *Let $B \subset C \subset M$, and let U be a coarse μ -connected component of $M \setminus B$. Then each coarse μ -connected component of $U \setminus C$ is a coarse μ -connected component of $M \setminus C$.*

PROOF. Each coarse μ -connected component V of $U \setminus C$ is contained in some coarse μ -connected component W of $M \setminus C$ (Remark 2.109-(i)). If $V \neq W$, then there are points, $y \in V$ and $z \in W \setminus V$, such that $d(y, z) \leq \mu$ (Remark 2.109-(iii)). Hence z is coarsely μ -connected to y in $M \setminus B$, and therefore in $M \setminus C$. So $z \in U$, and z is coarsely μ -connected to y in $U \setminus C$, giving $z \in V$, a contradiction. Thus $V = W$. \square

COROLLARY 2.115. *Let $A, B \subset M$, and let U be a coarse μ -connected component of $M \setminus B$ such that $U \cap A = \emptyset$. Then U is a coarse μ -connected component of $M \setminus (A \cup B)$.*

PROOF. Take $C = A \cup B$ in Lemma 2.114. \square

LEMMA 2.116. *Suppose that M is coarsely μ -connected. Let $\emptyset \neq B \subsetneq M$, and let U be a coarse μ -connected component of $M \setminus B$. Then there are points $x \in B$ and $y \in U$ such that $d(x, y) \leq \mu$.*

PROOF. Let $x_0 \in B$ and $y_0 \in U \setminus B$. Since M is coarsely μ -connected, there is a finite sequence $x_0 = z_0, z_1, \dots, z_k = y_0$ such that $d(z_{l-1}, z_l) \leq \mu$ for all $l \in \{1, \dots, k\}$. Let

$$p = \min\{l \in \{1, \dots, k\} \mid \{z_l, \dots, z_k\} \subset M \setminus B\}.$$

Then the statement holds with $x = z_{p-1}$ and $y = z_p$. \square

REMARK 2.117. Lemma 2.116 is a refinement of Remark 2.109-(iii).

COROLLARY 2.118. *Suppose that M is coarsely μ -connected. Let A be a μ -net of M , let $\emptyset \neq B \subset M$, and let U be a coarse μ -connected component of $M \setminus B$ such that $U \setminus \text{Pen}(B, 2\mu) \neq \emptyset$. Then $A \cap U \cap \text{Pen}(B, 3\mu) \neq \emptyset$.*

PROOF. The set $U \setminus \text{Pen}(B, 2\mu)$ is a coarse μ -connected component of $M \setminus \text{Pen}(B, 2\mu)$ by Lemma 2.114. Then, by Lemma 2.116, there are points $x \in \text{Pen}(B, 2\mu)$ and $y \in U \setminus \text{Pen}(B, 2\mu)$ with $d(x, y) \leq \mu$. Since $x \in M \setminus \text{Pen}(B, \mu)$ by the triangle inequality, we also get $x \in U$. Take some $z \in A$ such that $d(x, z) \leq \mu$. Since $z \in \text{Pen}(B, 3\mu) \setminus B$ by the triangle inequality, we have $z \in A \cap U \cap \text{Pen}(B, 3\mu)$. \square

COROLLARY 2.119. *Suppose that M is of coarse bounded geometry with coarse bound (R, Q_r) , and coarsely μ -connected for some $\mu \geq R$. Let B be a nonempty bounded subset of M . Then $M \setminus B$ has at most $Q_{\text{diam}(B)+3\mu}$ coarse μ -connected components that meet $M \setminus \text{Pen}(B, 2\mu)$.*

PROOF. Fix an (R, Q_r) -quasi-lattice Γ of M , and a point $x \in B$. By Corollary 2.118, the number of coarse μ -connected components of $M \setminus B$ that meet $M \setminus \text{Pen}(B, 2\mu)$ is bounded by

$$|\Gamma \cap \text{Pen}(B, 3\mu)| \leq |\Gamma \cap \overline{B}(x, \text{diam}(B) + 3\mu)| \leq Q_{\text{diam}(B)+3\mu}. \quad \square$$

COROLLARY 2.120. *Suppose that M is of coarse bounded geometry with coarse bound (R, Q_r) , and coarsely μ -connected for some $\mu \geq R$. For any nonempty bounded subset $B \subset M$, let C be the union of B and the bounded coarse μ -connected components of $M \setminus B$. Then C is bounded, and the coarse μ -connected components of $M \setminus C$ are the unbounded coarse μ -connected components of $M \setminus B$.*

PROOF. This is a consequence of Corollaries 2.119 and 2.115. \square

COROLLARY 2.121. *Suppose that M is of coarse bounded geometry with coarse bound (R, Q_r) , coarsely μ -connected for some $\mu \geq R$, and unbounded. Then the complement of each bounded subset of M has an unbounded coarse μ -connected component.*

PROOF. Take any bounded $B \subset M$. We can assume $B \neq \emptyset$ because M is unbounded. Thus, if all coarse μ -connected components of $M \setminus B$ were bounded, then M would be bounded by Corollary 2.120, a contradiction. \square

2.7.2. Coarse ends. Let $\mathcal{B}(M)$ (or simply \mathcal{B}) be the set of bounded subsets of M . For all $\mu > 0$ and $B \in \mathcal{B}$, let $\mathcal{U}_{\mu,B}(M)$ (or simply $\mathcal{U}_{\mu,B}$) denote the discrete space of unbounded coarse μ -connected components of $M \setminus B$. According to Remark 2.109-(i), for $B \subset C$ in \mathcal{B} , we get a map $\eta_{\mu,B,C} : \mathcal{U}_{\mu,B} \rightarrow \mathcal{U}_{\mu,C}$ determined by $\eta_{\mu,B,C}(U) \supset U$. These spaces $\mathcal{U}_{\mu,B}$ and maps $\eta_{\mu,B,C}$ form a projective system (over \mathcal{B} with “ \subset ”), denoted by $\{\mathcal{U}_{\mu,B}, \eta_{\mu,B,C}\}$.

DEFINITION 2.122. The projective limit of $\{\mathcal{U}_{\mu,B}, \eta_{\mu,B,C}\}$, denoted by $\mathcal{E}_\mu(M)$, is the space of μ -ends of M .

REMARK 2.123. $\mathcal{E}_\mu(M)$ is Hausdorff and totally disconnected because each space $\mathcal{U}_{\mu,B}$ is discrete.

Each $\mathbf{e} \in \mathcal{E}_\mu(M)$ can be described as a map defined on \mathcal{B} such that $\mathbf{e}(B) \in \mathcal{U}_{\mu,B}$ and $\mathbf{e}(B) \supset \mathbf{e}(C)$ if $B \subset C$. The maps $\eta_{\mu,B} : \mathcal{E}_\mu(M) \rightarrow \mathcal{U}_{\mu,B}$ satisfying the universal property of the inverse limit are given by $\eta_{\mu,B}(\mathbf{e}) = \mathbf{e}(B)$. Hence, for $B \subset C$ in \mathcal{B} ,

$$(2.26) \quad \eta_{\mu,B,C}(\mathbf{e}(B)) = \mathbf{e}(C).$$

REMARK 2.124. We have $\mathbf{e}(B) \cap \mathbf{e}(C) \neq \emptyset$ for all $\mathbf{e} \in \mathcal{E}_\mu(M)$ and $B, C \in \mathcal{B}$, because $\mathbf{e}(B) \cap \mathbf{e}(C) \supset \mathbf{e}(B \cup C)$.

For each $B \in \mathcal{B}$ and $U \in \mathcal{U}_{\mu,B}$, let

$$\mathcal{N}_\mu(B, U) = \{ \mathbf{e} \in \mathcal{E}_\mu(M) \mid \mathbf{e}(B) = U \}.$$

The family of the sets $\mathcal{N}_\mu(B, U)$ is a base of the topology of $\mathcal{E}_\mu(M)$. For any fixed $x_0 \in M$, an ultrametric d_{μ,x_0} inducing the topology of $\mathcal{E}_\mu(M)$ is given by

$$d_{\mu,x_0}(\mathbf{e}, \mathbf{f}) = \exp(-\sup\{n \in \mathbb{N} \mid \mathbf{e}(B(x_0, n)) = \mathbf{f}(B(x_0, n))\}).$$

REMARK 2.125. Since $\{B(x_0, n) \mid n \in \mathbb{N}\}$ is cofinal in \mathcal{B} , for each nested sequence $U_0 \supset U_1 \supset \dots$ with $U_n \in \mathcal{U}_{\mu,B(x_0,n)}$, there is a unique $\mathbf{e} \in \mathcal{E}_\mu(M)$ such that $U_n = \mathbf{e}(B(x_0, n))$ for all n .

REMARK 2.126. The Lipschitz equivalence class of d_{μ,x_0} is independent of x_0 ; in fact, for another point $x_1 \in M$ and an integer $N \geq d(x_0, x_1)$, we easily get $d_{\mu,x_1} \leq e^N d_{\mu,x_0}$.

According to Remark 2.109-(ii), for $0 < \mu \leq \nu$ and $B \in \mathcal{B}$, there is a map $\theta_{\mu,\nu,B} : \mathcal{U}_{\mu,B} \rightarrow \mathcal{U}_{\nu,B}$ determined by $\theta_{\mu,\nu,B}(U) \supset U$. The diagram

$$(2.27) \quad \begin{array}{ccc} \mathcal{U}_{\nu,C} & \xrightarrow{\eta_{\nu,B,C}} & \mathcal{U}_{\nu,B} \\ \theta_{\mu,\nu,C} \uparrow & & \uparrow \theta_{\mu,\nu,B} \\ \mathcal{U}_{\mu,C} & \xrightarrow{\eta_{\mu,B,C}} & \mathcal{U}_{\mu,B} \end{array}$$

is commutative for $B \subset C$ in \mathcal{B} because, for all $U \in \mathcal{U}_{\mu,B}$,

$$\begin{aligned} \eta_{\nu,B,C} \theta_{\mu,\nu,C}(U) &\supset \theta_{\mu,\nu,C}(U) \supset U, \\ \theta_{\mu,\nu,B} \eta_{\mu,B,C}(U) &\supset \eta_{\mu,B,C}(U) \supset U. \end{aligned}$$

Hence the maps $\theta_{\mu,\nu,B}$ induce a continuous map $\theta_{\mu,\nu} : \mathcal{E}_\mu(M) \rightarrow \mathcal{E}_\nu(M)$ determined by the condition $\eta_{\nu,B} \theta_{\mu,\nu} = \theta_{\mu,\nu,B} \eta_{\mu,B}$ for all $B \in \mathcal{B}$. Observe that

$$(2.28) \quad \theta_{\mu,\nu}(\mathbf{e})(B) = \eta_{\nu,B} \theta_{\mu,\nu}(\mathbf{e}) = \theta_{\mu,\nu,B} \eta_{\mu,B}(\mathbf{e}) = \theta_{\mu,\nu,B}(\mathbf{e}(B)).$$

On the other hand, like the commutativity of (2.27), it can be proved that

$$(2.29) \quad \theta_{\mu,\nu,B} \circ \theta_{\lambda,\mu,B} = \theta_{\lambda,\nu,B}$$

for $0 < \lambda \leq \mu \leq \nu$. Hence

$$\begin{aligned} \theta_{\mu,\nu} \theta_{\lambda,\mu}(\mathbf{e})(B) &= \theta_{\mu,\nu,B}(\theta_{\lambda,\mu}(\mathbf{e})(B)) = \theta_{\mu,\nu,B} \theta_{\lambda,\mu,B}(\mathbf{e}(B)) \\ &= \theta_{\lambda,\nu,B}(\mathbf{e}(B)) = \theta_{\lambda,\nu}(\mathbf{e})(B) \end{aligned}$$

by (2.28), giving $\theta_{\mu,\nu} \theta_{\lambda,\mu} = \theta_{\lambda,\nu}$. Thus the spaces $\mathcal{E}_\nu(M)$ and maps $\theta_{\mu,\nu}$ form a direct system of topological spaces, denoted by $\{\mathcal{E}_\mu(M), \theta_{\mu,\nu}\}$.

DEFINITION 2.127. The injective limit of $\{\mathcal{E}_\mu(M), \theta_{\mu,\nu}\}$, denoted by $\mathcal{E}_\infty(M)$, is called the space of *coarse ends* of M .

Let $\theta_\mu : \mathcal{E}_\mu(M) \rightarrow \mathcal{E}_\infty(M)$ be the maps that satisfy the universal property of the injective limit.

REMARK 2.128. It is easy to see that

$$\theta_{\mu,\nu} : (\mathcal{E}_\mu(M), d_{\mu,x_0}) \rightarrow (\mathcal{E}_\nu(M), d_{\nu,x_0})$$

is non-expanding for $\nu \geq \mu$.

REMARK 2.129. The definition of the space of coarse ends can be generalized to arbitrary coarse spaces as follows. With the terminology of [Roe03], for each entourage E of a coarse space X , define the *coarse E -connected components* like the above coarse μ -connected components by using the condition $(x, y) \in E$ instead of $d(x, y) \leq \mu$. Let \mathcal{B} be the family of bounded subsets of X (those $B \subset X$ so that $B \times B$ is an entourage). Then we can define $\mathcal{U}_{E,B}$ and $\mathcal{E}_E(X)$ like the above $\mathcal{U}_{\mu,B}$ and $\mathcal{E}_\mu(M)$. For entourages $E \subset F$, we get a continuous map $\theta_{E,F} : \mathcal{E}_E(X) \rightarrow \mathcal{E}_F(X)$ defined like the above $\theta_{\mu,\nu}$ for $\mu \leq \nu$. Then $\{\mathcal{E}_E(X), \theta_{E,F}\}$ is a direct system whose injective limit is $\mathcal{E}_\infty(X)$.

REMARK 2.130. Observe that the spaces $\mathcal{E}_\mu(M)$ and $\mathcal{E}_\infty(M)$ are nonempty if and only if M is unbounded.

PROPOSITION 2.131. *If M is of coarse bounded geometry with coarse bound (R, Q_r) , and coarsely μ -connected for some $\mu \geq R$, then $\mathcal{E}_\mu(X)$ is compact.*

PROOF. This holds because the spaces $\mathcal{U}_{\mu,B}$ are finite by Corollary 2.119. \square

LEMMA 2.132. *If M is of coarse bounded geometry with coarse bound (R, Q_r) , and coarsely μ -connected for some $\mu \geq R$, then $\eta_{\mu,B,C}$ is surjective for $\emptyset \neq B \subset C$ in \mathcal{B} .*

PROOF. Take any $U \in \mathcal{U}_{\mu,B}$. By Corollary 2.121, $U \setminus C$ has some unbounded coarse μ -connected component V . Then $V \in \mathcal{U}_{\mu,C}$ by Lemma 2.114, and $\eta_{\mu,B,C}(V) = U$. \square

COROLLARY 2.133. *If M is of coarse bounded geometry with coarse bound (R, Q_r) , and coarsely μ -connected for some $\mu \geq R$, then $\eta_{\mu,B}$ is surjective for $\emptyset \neq B \in \mathcal{B}$.*

PROOF. By inductively applying Lemma 2.132, it follows that for every $U \in \mathcal{U}_{\mu,B}$ there is a nested sequence $U = U_0 \supset U_1 \supset \dots$ such that $U_n \in \mathcal{U}_{\mu, \text{Pen}(B,n)}$ for all $n \in \mathbb{N}$. Since $\{\text{Pen}(B,n) \mid n \in \mathbb{N}\}$ is cofinal in \mathcal{B} , there is a unique $\mathbf{e} \in \mathcal{E}_\mu(M)$ such that $\mathbf{e}(\text{Pen}(B,n)) = U_n$ for all $n \in \mathbb{N}$; in particular, $U = \mathbf{e}(B) = \eta_{\mu,B}(\mathbf{e})$. \square

LEMMA 2.134. *If M is of coarse bounded geometry with coarse bound (R, Q_r) , and coarsely μ -connected for some $\mu \geq R$, then $\theta_{\mu,\nu,B}$ is surjective for nonempty sets $B \in \mathcal{B}$ and $\nu \geq \mu > 0$.*

PROOF. Every $V \in \mathcal{U}_{\nu,B}$ is union of coarse μ -connected components of $M \setminus B$ (Remark 2.109-(ii)). Moreover $M \setminus B$ has a finite number of coarse μ -connected components that meet $M \setminus \text{Pen}(B, 2\mu)$ by Corollary 2.119. Since V is unbounded, it follows that V contains some unbounded coarse μ -connected component U of $M \setminus B$. Thus $U \in \mathcal{U}_{\mu,B}$ and $\theta_{\mu,\nu,B}(U) = V$. \square

LEMMA 2.135. *If M is of coarse bounded geometry with coarse bound (R, Q_r) , and coarsely μ -connected for some $\mu \geq R$, then $\theta_{\mu,\nu} : \mathcal{E}_\mu(M) \rightarrow \mathcal{E}_\nu(M)$ has dense image for $\nu \geq \mu$.*

PROOF. This follows from Corollary 2.133, Lemma 2.134 and [Dug78, Appendix Two, Theorem 2.5-(2), p. 430]. \square

PROPOSITION 2.136. *If M is of coarse bounded geometry and coarsely connected, then $\mathcal{E}_\infty(M)$ is compact.*

PROOF. $\mathcal{E}_\mu(M)$ is compact for μ large enough by Proposition 2.131. Moreover it is also Hausdorff. So $\theta_{\mu,\nu}$ is surjective for μ large enough and $\nu \geq \mu$ by Lemma 2.135, obtaining that $\theta_\mu : \mathcal{E}_\mu(M) \rightarrow \mathcal{E}_\infty(M)$ is surjective. Therefore $\mathcal{E}_\infty(M)$ is compact. \square

LEMMA 2.137. (i) Suppose that M is the metric space of vertices of a connected graph. For $B \in \mathcal{B}$ and $N \in \mathbb{Z}^+$, let $\tilde{B} = \text{Pen}(B, \lceil \frac{N-1}{2} \rceil)$. Then every coarse N -connected component of $M \setminus \tilde{B}$ is contained in some coarse 1-connected component of $M \setminus B$.

(ii) Assume that M is a connected complete Riemannian manifold²². For every closed $B \in \mathcal{B}$ and $\mu, \varepsilon > 0$, let $\tilde{B} = \text{Pen}(B, \frac{\mu+\varepsilon}{2})$. Then every coarse μ -connected component of $M \setminus \tilde{B}$ is contained in some connected component of $M \setminus B$.

PROOF. (i) Let U be a coarse N -connected component of $M \setminus \tilde{B}$. For $x, y \in U$, there are points $x = z_0, z_1, \dots, z_k = y$ in U such that $d(z_{l-1}, z_l) \leq N$ for all $l \in \{1, \dots, k\}$. Then there are points, $z_{l-1} = u_0^l, u_1^l, \dots, u_{p_l}^l = z_l$, in M such that $p_l \leq N$ and $d(u_{q-1}^l, u_q^l) = 1$ for all $q \in \{1, \dots, p_l\}$. Observe that $u_q^l \in \text{Pen}(U, \lceil \frac{N-1}{2} \rceil) \subset M \setminus B$. So x is coarsely 1-connected to y in $M \setminus B$. It follows that U is a coarsely 1-connected subset of $M \setminus B$, and therefore it is contained in some coarsely 1-connected component of $M \setminus B$.

(ii) For $x, y \in U \in \mathcal{U}_{\mu, \tilde{B}}$, there are points $x = z_0, z_1, \dots, z_k = y$ in U such that $d(z_{l-1}, z_l) \leq \mu$ for all $l \in \{1, \dots, k\}$. Thus there is a smooth path α_l from z_{l-1} to z_l with length $< \mu + \varepsilon$. Each α_l is a path in $\text{Pen}(U, \frac{\mu+\varepsilon}{2}) \subset M \setminus B$, and so the product path $\alpha_1 \cdots \alpha_k$ joins x to y in $M \setminus B$. Thus U is a connected subset of $M \setminus B$, and hence it is contained in some connected component of $M \setminus B$. \square

PROPOSITION 2.138. If M is the metric space of vertices of a connected graph G that has finitely many edges abutting on each vertex, then $\mathcal{E}(G) \equiv \mathcal{E}_1(M) \approx \mathcal{E}_\infty(M)$, canonically.

PROOF. The definitions of $\mathcal{E}_1(M)$ and $\mathcal{E}(G)$ are canonically equivalent so it has to be shown that $\mathcal{E}_1(M) \approx \mathcal{E}_\infty(M)$, canonically, which will be a consequence of showing that $\theta_{1,N}$ is a homeomorphism for each $N \in \mathbb{Z}^+$. For every $B \in \mathcal{B}$, let $\tilde{B} = \text{Pen}(B, \lceil \frac{N-1}{2} \rceil)$. By Lemma 2.137-(i), a map $\xi_{N,B} : \mathcal{U}_{N,\tilde{B}} \rightarrow \mathcal{U}_{1,B}$ is determined by $\xi_{N,B}(U) \supset U$. Like in the case of (2.27), it can be easily checked that, for $B \subset C$ in \mathcal{B} , the diagrams

$$\begin{array}{ccccc} \mathcal{U}_{N,\tilde{C}} & \xrightarrow{\xi_{N,C}} & \mathcal{U}_{1,C} & \mathcal{U}_{N,\tilde{B}} & \xrightarrow{\xi_{N,B}} & \mathcal{U}_{1,B} & \mathcal{U}_{1,\tilde{B}} & \xrightarrow{\theta_{1,N,\tilde{B}}} & \mathcal{U}_{N,\tilde{B}} \\ \downarrow \eta_{N,\tilde{B},\tilde{C}} & & \downarrow \eta_{N,B,C} & \parallel & \downarrow \theta_{1,N,B} & & \parallel & & \downarrow \xi_{N,B} \\ \mathcal{U}_{N,\tilde{B}} & \xrightarrow{\xi_{N,B}} & \mathcal{U}_{1,B} & \mathcal{U}_{N,\tilde{B}} & \xrightarrow{\eta_{N,B,\tilde{B}}} & \mathcal{U}_{N,B} & \mathcal{U}_{1,\tilde{B}} & \xrightarrow{\eta_{1,B,\tilde{B}}} & \mathcal{U}_{1,B} \end{array}$$

are commutative. So the maps $\xi_{N,B}$ induce a continuous map $\xi_N : \mathcal{E}_N(M) \rightarrow \mathcal{E}_1(M)$ that is inverse of $\theta_{1,N}$. Thus $\theta_{1,N}$ is a homeomorphism. \square

PROPOSITION 2.139. If M is a connected complete Riemannian manifold, then $\mathcal{E}(M) \approx \mathcal{E}_\mu(M) \approx \mathcal{E}_\infty(M)$, canonically, for all $\mu > 0$.

²²In fact, the proof of this property applies to any complete path metric space, as well as the proof of Proposition 2.139.

PROOF. The proof is similar to that of Proposition 2.138, using Lemma 2.137-(ii). \square

2.7.3. Functoriality of the space of coarse ends. Let $f : M \rightarrow M'$ be a coarse map; in particular, it satisfies the condition of uniform expansiveness with some mapping s_r , which can be assumed to be non-decreasing. We have $f^{-1}(B') \in \mathcal{B}(M)$ for all $B' \in \mathcal{B}(M')$ because f is metric proper. For $x, y \in U \in \mathcal{U}_{\mu, f^{-1}(B')}$, there are points $x = z_0, z_1, \dots, z_k = y$ in U so that $d(z_{l-1}, z_l) \leq \mu$ for $l \in \{1, \dots, k\}$. Since $d'(f(z_l), f(z_{l-1})) \leq s_\mu$, we get that $f(U)$ is a coarsely s_μ -connected subset of $M' \setminus B'$. Hence $f(U)$ is contained in some coarse s_μ -connected component U' of $M' \setminus B'$. Moreover U' is unbounded; otherwise, $f^{-1}(f(U))$ is bounded because f is metrically proper, obtaining that U is bounded, a contradiction. Thus there is a map $f_{\mu, B'} : \mathcal{U}_{\mu, f^{-1}(B')}(M) \rightarrow \mathcal{U}_{s_\mu, B'}(M')$ determined by $f_{\mu, B'}(U) \supset f(U)$. Since $B \subset f^{-1}(f(B))$ for all $B \in \mathcal{B}(M)$, and $f(B) \in \mathcal{B}(M')$ by the uniform expansiveness of f , the set $\{f^{-1}(B') \mid B' \in \mathcal{B}(M')\}$ is cofinal in \mathcal{B} . Hence the maps $f_{\mu, B'}$ induce a continuous map $f_\mu : \mathcal{E}_\mu(M) \rightarrow \mathcal{E}_{s_\mu}(M')$, determined by the condition $\eta_{\mu, B'} \circ f_\mu = f_{\mu, B'} \circ \eta_{\mu, f^{-1}(B')}$ for all $B' \in \mathcal{B}(M')$. Thus

$$(2.30) \quad f_\mu(\mathbf{e})(B') = \eta_{\mu, B'} \circ f_\mu(\mathbf{e}) = f_{\mu, B'} \circ \eta_{\mu, f^{-1}(B')}(\mathbf{e}) = f_{\mu, B'}(\mathbf{e}(f^{-1}(B')))$$

for all $B' \in \mathcal{B}(M')$. As in the case of (2.27), it is easy to check that the diagram

$$\begin{array}{ccc} \mathcal{U}_{\nu, f^{-1}(B')}(M) & \xrightarrow{f_\nu} & \mathcal{U}_{s_\nu, B'}(M') \\ \theta_{\mu, \nu, f^{-1}(B')} \uparrow & & \uparrow \theta_{s_\mu, s_\nu, B'} \\ \mathcal{U}_{\mu, f^{-1}(B')}(M) & \xrightarrow{f_\mu} & \mathcal{U}_{s_\mu, B'}(M') \end{array}$$

is commutative for $0 < \mu < \nu$, and thus the diagram

$$\begin{array}{ccc} \mathcal{E}_\nu(M) & \xrightarrow{f_\nu} & \mathcal{E}_{s_\nu}(M') \\ \theta_{\mu, \nu} \uparrow & & \uparrow \theta_{s_\mu, s_\nu} \\ \mathcal{E}_\mu(M) & \xrightarrow{f_\mu} & \mathcal{E}_{s_\mu}(M') \end{array}.$$

is also commutative. So the maps f_μ induce a continuous map $f_\infty : \mathcal{E}_\infty(M) \rightarrow \mathcal{E}_\infty(M')$.

LEMMA 2.140. f_∞ is independent of the choice of s_r .

PROOF. Let $\bar{s}_r \geq s_r$ for each $r \geq 0$, and let

$$\begin{aligned} \bar{f}_{\mu, B'} : \mathcal{U}_{\mu, f^{-1}(B')}(M) &\rightarrow \mathcal{U}_{\bar{s}_\mu, B'}(M'), \\ \bar{f}_\mu : \mathcal{E}_\mu(M) &\rightarrow \mathcal{E}_{\bar{s}_\mu}(M'), \quad \bar{f}_\infty : \mathcal{E}_\infty(M) \rightarrow \mathcal{E}_\infty(M') \end{aligned}$$

be the maps induced by f and \bar{s}_r . For $\mu > 0$ and $B' \in \mathcal{B}(M')$, the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{U}_{\mu, f^{-1}(B')}(M) & \xrightarrow{\bar{f}_{\mu, B'}} & \mathcal{U}_{\bar{s}_\mu, B'}(M') \\ \parallel & & \uparrow \theta_{s_\mu, \bar{s}_\mu, B'} \\ \mathcal{U}_{\mu, f^{-1}(B')}(M) & \xrightarrow{f_{\mu, B'}} & \mathcal{U}_{s_\mu, B'}(M') \end{array}$$

follows like in the case of (2.27), obtaining the commutativity of

$$\begin{array}{ccc} \mathcal{E}_\mu(M) & \xrightarrow{\bar{f}_\mu} & \mathcal{E}_{\bar{s}_\mu}(M') \\ \parallel & & \uparrow \theta_{s_\mu, \bar{s}_\mu} \\ \mathcal{E}_\mu(M) & \xrightarrow{f_\mu} & \mathcal{E}_{s_\mu}(M') . \end{array}$$

Therefore $\bar{f}_\infty = f_\infty$. \square

PROPOSITION 2.141. *If $g : M \rightarrow M'$ is another coarse map close to f , then $g_\infty = f_\infty$.*

PROOF. Take some $c \geq 0$ such that f and g are c -close. We can suppose that g also satisfies the condition of uniform expansiveness with s_r . Let $\mu > 0$, $B \in \mathcal{B}(M)$, $B' \in \mathcal{B}(M')$, $U \in \mathcal{U}_{\mu, B}(M)$, $V \in \mathcal{U}_{\mu, f^{-1}(B')}(M)$ and $W \in \mathcal{U}_{\mu, g^{-1}(B')}(M)$ such that $f^{-1}(B') \cup g^{-1}(B') \subset B$ and $U \subset V \cap W$. Therefore $\eta_{\mu, f^{-1}(B'), B}(U) = V$ and $\eta_{\mu, g^{-1}(B'), B}(U) = W$. We know that $f(V)$ and $g(W)$ are coarsely s_μ -connected subsets of $M' \setminus B'$. By Remark 2.109-(iv) and since $d'(f(x), g(x)) \leq c$ for all $x \in U$, it follows that $f(V) \cup g(W)$ is a coarsely \bar{s}_μ -connected subset of $M' \setminus B'$, where $\bar{s}_\mu = \max\{s_\mu, c\}$. Hence $f(V)$ and $g(W)$ are contained in the same coarse \bar{s}_μ -connected component of $M' \setminus B'$. This shows that the diagram

$$\begin{array}{ccccc} \mathcal{U}_{\mu, f^{-1}(B')}(M) & \xrightarrow{f_{\mu, B'}} & \mathcal{U}_{s_\mu, B'}(M') & \xrightarrow{\theta_{s_\mu, \bar{s}_\mu, B'}} & \mathcal{U}_{\bar{s}_\mu, B'}(M') \\ \eta_{\mu, f^{-1}(B'), B} \uparrow & & & & \uparrow \theta_{s_\mu, \bar{s}_\mu, B'} \\ \mathcal{U}_{\mu, B}(M) & \xrightarrow{\eta_{\mu, g^{-1}(B'), B}} & \mathcal{U}_{\mu, g^{-1}(B')}(M) & \xrightarrow{g_{\mu, B'}} & \mathcal{U}_{s_\mu, B'}(M') \end{array}$$

is commutative. So the diagram

$$\begin{array}{ccc} \mathcal{E}_{s_\mu}(M') & \xrightarrow{\theta_{s_\mu, \bar{s}_\mu}} & \mathcal{E}_{\bar{s}_\mu}(M') \\ f_\mu \uparrow & & \uparrow \theta_{s_\mu, \bar{s}_\mu} \\ \mathcal{E}_\mu(M) & \xrightarrow{g_\mu} & \mathcal{E}_{s_\mu}(M') \end{array}$$

is also commutative, and $f_\infty = g_\infty$ obtains. \square

PROPOSITION 2.142. *A functor \mathcal{E}_∞ of the metric coarse category to the category of continuous maps between topological spaces is defined by $\mathcal{E}_\infty([M]) = \mathcal{E}_\infty(M)$ and $\mathcal{E}_\infty([f]) = f_\infty$.*

PROOF. Obviously, $(\text{id}_M)_\mu = \text{id}_{\mathcal{E}_\mu(M)}$ for all $\mu > 0$, and therefore $(\text{id}_M)_\infty = \text{id}_{\mathcal{E}_\infty(M)}$. Suppose that $f' : M' \rightarrow M''$ satisfies the condition of uniform expansiveness with a mapping $s'_{r'}$. Then $f'f$ satisfies the condition of uniform expansiveness with the mapping s'_{s_r} , and, like in the case of (2.27), we get that the composite

$$\mathcal{U}_{\mu, f^{-1}(f'^{-1}(B''))}(M) \xrightarrow{f_{\mu, f'^{-1}(B'')}} \mathcal{U}_{s_\mu, f'^{-1}(B'')}(M') \xrightarrow{f'_{s_\mu, B''}} \mathcal{U}_{s'_{s_\mu}, B''}(M'')$$

equals $(f'f)_{\mu, B''}$ for all $\mu > 0$ and $B'' \in \mathcal{B}(M'')$. So the composite

$$\mathcal{E}_\mu(M) \xrightarrow{f_\mu} \mathcal{E}_{s_\mu}(M') \xrightarrow{f'_{s_\mu}} \mathcal{E}_{s'_{s_\mu}}(M'')$$

equals $(f'f)_\mu$ for all $\mu > 0$, obtaining $f'_\infty f_\infty = (f'f)_\infty$. This shows that $M \mapsto \mathcal{E}_\infty(M)$ and $f \mapsto f_\infty$ defines a functor of the category of coarse maps between metric spaces to the category of continuous maps between topological spaces. Then $\mathcal{E}_\infty(M)$ depends only on $[M]$. Furthermore f_∞ depends only on $[f]$ by Lemma 2.140 and Proposition 2.141. \square

REMARK 2.143. Continuing with the ideas of Remark 2.129, the functor \mathcal{E}_∞ can be obviously extended to the whole coarse category.

COROLLARY 2.144. *If $f : M \rightarrow M'$ is a coarse equivalence, then f_∞ is a homeomorphism.*

COROLLARY 2.145. *Coarsely quasi-isometric metric spaces have homeomorphic spaces of coarse ends.*

2.7.4. Coarse end space of a class of metric spaces. The following result extends a well known theorem for finitely generated groups (Example 2.45) [Geo08, Theorem 13.5.7].

THEOREM 2.146. *Assume that M is of coarse bounded geometry, coarsely quasi-convex and coarsely quasi-symmetric. Then, either $|\mathcal{E}_\infty(M)| \leq 2$, or $\mathcal{E}_\infty(M)$ is a Cantor space.*

PROOF. By Corollaries 2.56 and 2.145, Propositions 2.63 and 2.69, and Example 2.51, we can assume that M is the metric space of vertices of a connected graph of finite type. Thus $\mathcal{E}_1(M) \approx \mathcal{E}_\infty(M)$ by Proposition 2.138. Suppose that $|\mathcal{E}_1(M)| \geq 3$, and let us prove that $\mathcal{E}_1(M)$ is a Cantor space. Take three distinct elements $\mathbf{e}_k \in \mathcal{E}_1(M)$, $k \in \{1, 2, 3\}$. Since $\mathcal{E}_1(M)$ is Hausdorff, totally disconnected and compact (Propositions 2.136 and 2.138), it is enough to prove that $|\mathcal{N}_1(B, U)| \geq 2$ for any basic open set $\mathcal{N}_1(B, U)$ of $\mathcal{E}_1(M)$ (Section 2.7.2). We can assume that B is coarsely 1-connected and the sets $\mathbf{e}_k(B)$ are distinct.

By Propositions 2.26 and 2.39, there is a transitive set \mathcal{T} of equi-rough transformations of M , and let (s_r, c) be a common rough equivalence distortion of all maps in \mathcal{T} . We can suppose that $s_r \uparrow \infty$ as $r \rightarrow \infty$, and $s_n, c \in \mathbb{N}$ for all $n \in \mathbb{N}$. Let $N = \max\{s_{s_1}, c\} \geq 1$ and $\tilde{B} = \text{Pen}(B, \lceil \frac{N-1}{2} \rceil)$.

CLAIM 4. For each $f \in \mathcal{T}$, the sets $f_1(\mathbf{e}_k)(\text{Pen}(f(\tilde{B}), c))$ are distinct.

Given any $f \in \mathcal{T}$, let $g : M \rightarrow M$ be a rough equivalence with rough distortion s_r such that gf and fg are c -close to id_M . Let $C = \text{Pen}(\tilde{B}, c)$. We have $(gf)^{-1}(\tilde{B}) \subset C$ because, if $x \in (gf)^{-1}(\tilde{B})$, then $gf(x) \in \tilde{B}$ and $d(x, gf(x)) \leq c$. Then the diagram

$$\begin{array}{ccccc}
 \mathcal{U}_{1, \tilde{B}} & \xrightarrow{\eta_{1, B, \tilde{B}}} & \mathcal{U}_{1, B} & & \\
 \parallel & & \uparrow \xi_{N, B} & & \\
 \mathcal{U}_{1, \tilde{B}} & \xrightarrow{\theta_{1, N, \tilde{B}}} & \mathcal{U}_{N, \tilde{B}} & \xleftarrow{\theta_{s_{s_1}, N, \tilde{B}}} & \mathcal{U}_{s_{s_1}, \tilde{B}} \\
 \uparrow \eta_{1, \tilde{B}, C} & & & & \uparrow g_{1, \tilde{B}} \\
 \mathcal{U}_{1, C} & \xrightarrow{\eta_{1, (gf)^{-1}(\tilde{B}), C}} & \mathcal{U}_{1, (gf)^{-1}(\tilde{B})}(M) & \xrightarrow{f_{1, g^{-1}(\tilde{B})}} & \mathcal{U}_{s_1, g^{-1}(\tilde{B})}
 \end{array}$$

is commutative according to the proofs of Propositions 2.138, 2.141 and 2.142. Hence, by (2.26),

$$\begin{aligned} \xi_{N,B} \circ \theta_{s_1,N,\tilde{B}} \circ g_{1,\tilde{B}} \circ f_{1,g^{-1}(\tilde{B})} \circ \eta_{1,(gf)^{-1}(\tilde{B}),C}(\mathbf{e}_k(C)) \\ = \xi_{N,B} \circ \theta_{1,N,\tilde{B}} \circ \eta_{1,\tilde{B},C}(\mathbf{e}_k(C)) = \xi_{N,B} \circ \theta_{1,N,\tilde{B}}(\mathbf{e}_k(\tilde{B})) \\ = \eta_{1,B,\tilde{B}}(\mathbf{e}_k(\tilde{B})) = \mathbf{e}_k(B), \end{aligned}$$

which are distinct sets for $k \in \{1, 2, 3\}$. Thus the sets

$$\begin{aligned} f_{1,g^{-1}(\tilde{B})} \circ \eta_{1,(gf)^{-1}(\tilde{B}),C}(\mathbf{e}_k(C)) = f_{1,(gf)^{-1}(\tilde{B})}(\mathbf{e}_k((gf)^{-1}(\tilde{B}))) \\ = f_1(\mathbf{e}_k)(\mathbf{e}_k(g^{-1}(\tilde{B}))) \end{aligned}$$

are also distinct, where we have used (2.26) and (2.30). On the other hand, $g^{-1}(\tilde{B}) \subset \text{Pen}(f(\tilde{B}), c)$ because, if $x \in g^{-1}(\tilde{B})$, then $fg(x) \in f(\tilde{B})$ and $d(fg(x), x) \leq c$. Thus Claim 4 follows because

$$f_1(\mathbf{e}_k)(\mathbf{e}_k(g^{-1}(\tilde{B}))) = \eta_{1,g^{-1}(\tilde{B}), \text{Pen}(f(\tilde{B}), c)} \circ f_1(\mathbf{e}_k)(\mathbf{e}_k(\text{Pen}(f(\tilde{B}), c))).$$

Take any integer $R \geq \text{diam } B$. Since U is unbounded and \mathcal{T} transitive, we can fix some $f \in \mathcal{T}$ such that there is some $x \in U \cap f(\tilde{B})$ with

$$d(x, B) > c + s_{R+N} + N + 1.$$

Since $\text{diam } \tilde{B} \leq R + N$, we have $\text{diam } f(\tilde{B}) \leq s_{R+N}$, obtaining

$$(2.31) \quad d(f(\tilde{B}), B) > c + N + 1.$$

Let

$$\begin{aligned} B' = \text{Pen}(f(\tilde{B}), c), \quad B'' = \text{Pen}(f(\tilde{B}), c + \lceil \frac{N-1}{2} \rceil), \\ V_k = f_1(\mathbf{e}_k)(B') \in \mathcal{U}_{s_1, B'}, \quad W_k = f_1(\mathbf{e}_k)(B'') \in \mathcal{U}_{s_1, B''}. \end{aligned}$$

Using the notation of the proof of Proposition 2.138, the composite

$$\mathcal{U}_{s_1, B''} \xrightarrow{\theta_{s_1, N, B''}} \mathcal{U}_{N, B''} \xrightarrow{\xi_{N, B'}} \mathcal{U}_{1, B'}$$

is defined because $N \geq s_1$. So each W_k is contained in some $U'_k \in \mathcal{U}_{1, B'}$. Moreover $U'_k \subset V_k$ by Remark 2.109-(ii) because $W_k \subset V_k$; in particular, the sets U'_k are disjoint from each other by Claim 4.

CLAIM 5. U meets all sets U'_k .

Since B is coarsely 1-connected, $\text{Pen}(B'', 1)$ is coarsely s_1 -connected. Moreover $\text{Pen}(B'', 1)$ is disjoint from \tilde{B} by (2.31). So $\text{Pen}(B'', 1)$ is contained in some coarse 1-connected component of $M \setminus B$ by Lemma 2.137-(i) and Remark 2.109-(i) because $N \geq s_1$. Since $f(B)$ meets U , we get $\text{Pen}(B'', 1) \subset U$. So U meets every W_k by Lemma 2.116, and therefore it also meets every U'_k , showing Claim 5.

CLAIM 6. \tilde{B} meets at most one of the sets U'_k .

Indeed, suppose that \tilde{B} meets two of the sets U'_k , say U'_1 and U'_2 . Since $U'_1, U'_2 \in \mathcal{U}_{1, B'}$ and \tilde{B} is coarsely 1-connected and disjoint from B' by (2.31), it follows that $U'_1 \cup U'_2 \cup \tilde{B}$ is an unbounded coarsely 1-connected subset of $M \setminus B'$. Therefore $U'_1 = U'_2$, a contradiction, confirming Claim 6.

According to Claim 6, assume from now on that $U'_1, U'_2 \subset M \setminus \tilde{B}$. Since these subsets are coarsely 1-connected, they are contained in coarse 1-connected components of $M \setminus B$ (Remark 2.109-(i)). Then $U'_1, U'_2 \subset U$ by Claim 5. By Corollary 2.115, $U'_1, U'_2 \in \mathcal{U}_{1, B \cup B'}$, and, by Corollary 2.133, there are $\mathbf{e}'_1, \mathbf{e}'_2 \in \mathcal{E}_1(M)$ such that $\mathbf{e}'_k(B \cup B') = U'_k$ for $k \in \{1, 2\}$. So $\mathbf{e}'_1 \neq \mathbf{e}'_2$ and $\mathbf{e}'_k(B) \supset \mathbf{e}'_k(B \cup B') = U'_k$, obtaining $\mathbf{e}'_k(B) = U$ because $U'_k \subset U$. Thus $\mathbf{e}'_1, \mathbf{e}'_2 \in \mathcal{N}_1(B, U)$, showing that $|\mathcal{N}_1(B, U)| \geq 2$. \square

2.8. Higson Compactification

2.8.1. Compactifications. Recall that a *compactification* of a topological space X is a pair (Y, h) consisting of a compact Hausdorff²³ space Y and an embedding $h : X \rightarrow Y$ with dense image. The subspace $Y \setminus h(X) \subset Y$ is called the *corona* of the compactification. Usually, the notation is simplified by assuming that $X \subset Y$ and h is the inclusion map, which is omitted from the notation; in particular, it will be simply said that X is open in Y when h is an open map. A typical notation is \overline{X} for a compactification and ∂X for the corona, or X^γ for the compactification and γX for the corona, specially when γ refers to some kind of compactification of a class of spaces.

The space X admits a compactification if and only if it is Tychonov. Moreover X is open in some compactification if and only if it is locally compact and Hausdorff; in this case, X is open in any compactification.

Two compactifications of X , (Y, h) and (Y', h') , are *equivalent* when there is a homeomorphism $\phi : Y' \rightarrow Y$ so that $\phi h' = h$. The term “compactification” will refer to an equivalence classes of compactifications. In this sense, the set²⁴ of compactifications has a partial order relation, “ \leq ”, defined by declaring $(Y, h) \leq (Y', h')$ if there is a continuous $\pi : Y' \rightarrow Y$ so that $\pi h' = h$.

For a locally compact Hausdorff space X , let $C_b(X)$ denote the commutative C^* algebra of bounded \mathbb{C} -valued continuous functions on X with the supremum norm, and let $C_0(X) \subset C_b(X)$ be the closed involutive ideal of continuous functions that vanish at infinity²⁵. The closed subalgebra of constant functions on X may be canonically identified to \mathbb{C} .

The Gelfand-Naimark theorem estates that the assignment $X \mapsto C_b(X)$ defines a one-to-one correspondence between the (homeomorphism classes of) compact Hausdorff spaces and the (isomorphism classes of) commutative C^* algebras with unit. The compact Hausdorff space $\Delta(\mathcal{A})$ that corresponds to each unital commutative C^* algebra \mathcal{A} is the space of characters $\mathcal{A} \rightarrow \mathbb{C}$ with the topology of pointwise convergence. For a compact Hausdorff space X , a canonical homeomorphism $h : X \rightarrow \Delta(C_b(X))$ is given by evaluation: $h(x)(f) = f(x)$ for all $x \in X$ and $f \in C_b(X)$.

With more generality, when X is a locally compact Hausdorff space X , the assignment $(Y, h) \mapsto h^*C(Y)$ is a one-to-one correspondence between (equivalence classes) of Hausdorff compactifications of X and unital closed involutive subalgebras

²³We only consider Hausdorff compactifications.

²⁴All compactifications of X are $\leq X^\beta$, where X^β is the Stone-Ćech compactification. Thus we can assume that they are quotients of X^β , and therefore they form a set.

²⁵Recall that a function $f : X \rightarrow \mathbb{C}$ *vanishes at infinity* when, for all $\varepsilon > 0$, there is a compact $K \subset X$ so that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$.

of $C_b(X)$ that generate the topology (in the sense that compact sets can be separated from points by functions in the algebra). For each subalgebra $\mathcal{A} \subset C(X)$ of the above type, the corresponding compactification is $(\Delta(\mathcal{A}), h)$, where $h : X \rightarrow \Delta(\mathcal{A})$ is defined by evaluation at each point of X , as before. For instance, $C_b(X)$ corresponds to the Stone-Ćech compactification X^β , and $\mathbb{C} + C_0(X)$ corresponds to the one-point compactification X^* .

EXAMPLE 2.147. Consider the notation of Section 2.7. If M is of coarse bounded geometry and coarsely connected, then $\mathcal{E}_\infty(M)$ is compact (Proposition 2.136). If moreover M is proper, then $\mathcal{E}_\infty(M)$ is the corona of a compactification of M , which can be seen as follows. To show this property, let us prove first that $\mathcal{E}_\mu(M)$ is the corona of a compactification of M for $\mu > 0$ large enough, like the usual space of ends of a manifold or a graph. The space $\mathcal{E}_\mu(M)$ is compact for $\mu > 0$ large enough (Proposition 2.131). Then let $\overline{M}_\mu = M \cup \mathcal{E}_\mu(M)$ with the topology so that the inclusion map $M \hookrightarrow \overline{M}_\mu$ is an open embedding, and a base of neighborhoods in \overline{M}_μ of any $\mathbf{e} \in \mathcal{E}_\mu(M)$ is given by the sets $\mathbf{e}(B)$ for $B \in \mathcal{B}(M)$. Using Corollary 2.119, it can be easily seen that \overline{M}_μ is compact, and it is obvious that M is dense in \overline{M}_μ . Now, for $\mu < \nu$ large enough, the continuous map $\theta_{\mu,\nu} : \mathcal{E}_\mu(M) \rightarrow \mathcal{E}_\nu(M)$ is surjective because it has dense image (Lemma 2.135) and these spaces are compact and Hausdorff. The combination of $\theta_{\mu,\nu}$ and id_M , denoted by $\bar{\theta}_{\mu,\nu} : \overline{M}_\mu \rightarrow \overline{M}_\nu$, is continuous by (2.28) and the definition of $\theta_{\mu,\nu,B}$; thus $\overline{M}_\nu \leq \overline{M}_\mu$. Moreover $\{\overline{M}_\mu, \bar{\theta}_{\mu,\nu}\}$ is a direct system of topological spaces because so is $\{\mathcal{E}_\mu(M), \theta_{\mu,\nu}\}$. Since $\mathcal{E}_\infty(M)$ is the injective limit of $\{\mathcal{E}_\mu(M), \theta_{\mu,\nu}\}$, it easily follows that the injective limit of $\{\overline{M}_\mu, \bar{\theta}_{\mu,\nu}\}$ is a compactification \overline{M}_∞ of M with corona $\mathcal{E}_\infty(M)$.

LEMMA 2.148. *Let X^γ and X'^γ be compactifications of locally compact Hausdorff spaces X and X' , with coronas γX and $\gamma X'$, respectively. Let $\phi : X \rightarrow X'$ and $\psi : X' \rightarrow X$ be (possibly non-continuous) maps such that:*

- (i) *ϕ and ψ have extensions, $\phi^\gamma : X^\gamma \rightarrow X'^\gamma$ and $\psi^\gamma : X'^\gamma \rightarrow X^\gamma$, which are continuous at the points of γX and $\gamma X'$, respectively; and*
- (ii) *ϕ^γ and ψ^γ restrict to respective homeomorphisms, $\gamma\phi : \gamma X \rightarrow \gamma X'$ and $\gamma\psi : \gamma X' \rightarrow \gamma X$, which are inverse of each other.*

Then there is a bijection $\overline{X} \leftrightarrow \overline{X}'$ between the set of compactifications $\overline{X} \leq X^\gamma$, with coronas ∂X , and the set of compactifications $\overline{X}' \leq X'^\gamma$, with coronas $\partial X'$, such that:

- (a) *ϕ and ψ have extensions, $\bar{\phi} : \overline{X} \rightarrow \overline{X}'$ and $\bar{\psi} : \overline{X}' \rightarrow \overline{X}$, which are continuous at the points of ∂X and $\partial X'$, respectively; and*
- (b) *$\bar{\phi}$ and $\bar{\psi}$ restrict to respective homeomorphisms, $\partial\phi : \partial X \rightarrow \partial X'$ and $\partial\psi : \partial X' \rightarrow \partial X$, which are inverse of each other.*

PROOF. Take a compactification $\overline{X} \leq X^\gamma$, with corona ∂X . Thus id_M has a continuous extension $\pi : \gamma X \rightarrow \partial X$, which is an identification. Then the restriction $\pi : X^\gamma \rightarrow \partial X$ is also an identification. Let R be the equivalence relation on γX whose equivalence classes are the fibers of π (thus $\partial X \equiv \gamma X/R$), let R' be the equivalence relation on $\gamma X'$ that corresponds to R via $\gamma\phi$, and let $\partial X' = \gamma X'/R'$. Then $\partial\phi$ induces a homeomorphism $\partial'\phi : \partial'X \rightarrow \partial'Y$. Extend R' to X'^γ so that each point of X' is only equivalent to itself, and let $\overline{X}' = X'^\gamma/R'$. Let $\pi' : X'^\gamma \rightarrow \overline{X}'$ be the quotient map, whose restriction $\pi' : \gamma X' \rightarrow \partial X'$ is also the quotient map.

Using that X' is open in $\overline{X'}$, it easily follows that the restriction of $\pi' : X' \rightarrow \overline{X'}$ is an embedding with dense image. In this way, $\overline{X'}$ becomes a compactification of X' , with corona $\partial X'$, satisfying $\overline{X'} \leq X'^\gamma$. This defines the stated mapping $\overline{X} \mapsto \overline{X'}$.

The above notation will be continued in the whole proof. Extend R to X^γ so that each point of X is only equivalent to itself. Then, like in the case of R' , the space X^γ/R is a compactification of X . Moreover the canonical map $X^\gamma/R \rightarrow \overline{X}$ is a continuous bijection between compact Hausdorff spaces, and thus it is a homeomorphism, showing that the compactifications X^γ/R and \overline{X} are equivalent. Since R corresponds to R' via $\gamma\psi = (\gamma\phi)^{-1}$, it follows that \overline{X} corresponds to $\overline{X'}$ by the mapping defined by ψ in the same way. Hence the mapping defined by ψ is left inverse of the mapping defined by ϕ . Reversing also the roles played by ϕ and ψ , it follows that the mapping $\overline{X} \mapsto \overline{X'}$ of the statement is bijective.

Since ϕ^γ is compatible with R and R' , it induces a map $\bar{\phi} : \overline{X} \rightarrow \overline{X'}$. Let us show that $\bar{\phi}$ is continuous at every $\mathbf{e} \in \partial X$. Let V' be a neighborhood of $\mathbf{e}' := \bar{\phi}(\mathbf{e})$ in $\overline{X'}$. The set $\tilde{V}' := \pi'^{-1}(V')$ is an R' -saturated neighborhood of $\pi'^{-1}(\mathbf{e}')$ in X'^γ . So, by (i), $\tilde{V} := (\phi^\gamma)^{-1}(\tilde{V}') = \pi^{-1}(\bar{\phi}^{-1}(V'))$ is an R -saturated neighborhood of $\tilde{V} := (\phi^\gamma)^{-1}(\pi'^{-1}(\mathbf{e}')) = \pi^{-1}(\bar{\phi}^{-1}(\mathbf{e}'))$ in X^γ . Hence $\pi(\tilde{V}) = \bar{\phi}^{-1}(V')$ is a neighborhood of $\bar{\phi}^{-1}(\mathbf{e}')$ in \overline{X} . Similarly, ψ induces a map $\bar{\psi} : \overline{X'} \rightarrow \overline{X}$, which is continuous at the points of $\partial X'$. This completes the proof of (a).

Property (b) follows directly from (ii) because $\partial\phi$ and $\partial\psi$ are induced by $\gamma\phi$ and $\gamma\psi$, respectively. \square

2.8.2. Higson compactification. Suppose that the metric space M is proper. For $R > 0$, the R -variation a function $f : M \rightarrow \mathbb{C}$ is the function $\mathbf{V}_R f : M \rightarrow \mathbb{C}$ be given by

$$\mathbf{V}_R f(x) = \sup\{|f(x) - f(y)| \mid d(x, y) < R\}.$$

It is said that $f : M \rightarrow \mathbb{C}$ is a *Higson function* if it is bounded and $\mathbf{V}_R f$ vanishes at infinity for all $R > 0$. The continuous Higson functions on M form a unital closed involutive subalgebra $C_\nu(X) \subset C_b(X)$ that generates the topology. The compactification of M that corresponds to $C_\nu(X)$ is called the *Higson compactification*, and $\nu M = M^\nu \setminus M$ is called the *Higson corona* of M .

REMARK 2.149. (i) The construction of the Higson corona can be extended to the case where M is not proper in the following way. The (possibly non-continuous) Higson functions form a unital closed involutive subalgebra $\mathcal{B}_\nu(M)$ of the commutative C^* algebra of \mathbb{C} -valued bounded functions on M with the supremum norm. Now, it is said that a function $f : M \rightarrow \mathbb{C}$ *vanishes at infinity* if, for all bounded subset $B \subset M$, there is some $r > 0$ such that $|f| < r$ on $M \setminus B$. The functions vanishing at infinity form a closed involutive ideal $\mathcal{B}_0 \subset \mathcal{B}_\nu(M)$. Then the Higson corona νM is the compact Hausdorff space that corresponds to the unital commutative C^* algebra $\mathcal{B}_\nu(M)/\mathcal{B}_0(M)$; this C^* algebra is isomorphic to $C_\nu(M)/C_0(M) \cong C(\nu M)$ when M is proper [Roe03, Lemma 2.40].

- (i) In fact, the Higson compactification can be defined for arbitrary proper coarse spaces, and the Higson corona can be defined for all coarse spaces [Roe03, Section 2.3].

PROPOSITION 2.150 (Álvarez-Candel [ALC11, Corollary 4.14, Proposition 4.15 and Theorem 4.16]). *The following properties hold for maps $\phi, \psi : M \rightarrow M'$ between proper metric spaces:*

- (i) ϕ is coarse if and only if it has an extension $\phi^\nu : M^\nu \rightarrow M'^\nu$ that is continuous at the points of νM and such that $\phi^\nu(\nu M) \subset \nu M'$. In particular, ϕ^ν restricts to a continuous map $\nu\phi : \nu M \rightarrow \nu M'$.
- (ii) ϕ is close to ψ if and only if the extensions ϕ^ν and ψ^ν , given by (i), are equal on νM .
- (iii) ϕ is a coarse equivalence if and only if it satisfies the conditions of (i) and $\nu\phi : \nu M \rightarrow \nu M'$ is a homeomorphism.

PROPOSITION 2.151. *If $\phi : M \rightarrow M'$ is rough map, then $\nu\phi : \nu M \rightarrow \nu M'$ is an embedding whose image is $\text{Cl}_{M^\nu}(\phi(M)) \cap \nu M'$.*

PROOF. By Corollary 2.38 and Proposition 2.150-(iii), it can be assumed that M is a metric subspace of M' , and ϕ is the inclusion map $M \hookrightarrow M'$. Since the Higson coronas are compact Hausdorff metric spaces and $\nu\phi$ is continuous (Proposition 2.150-(i)), it is enough to prove that $\nu\phi$ is injective.

Let $e_0 \neq e_1$ in νM . Take open subsets $V_0, V_1 \subset M^\nu$ such that $e_i \in V_i$ ($i \in \{0, 1\}$) and $\text{Cl}_{M^\nu}(V_0) \cap \text{Cl}_{M^\nu}(V_1) = \emptyset$. Fix any $x_0 \in M \setminus \text{Cl}_{M^\nu}(V_0 \cup V_1)$, and let $B_R = B_M(x_0, R)$ for each $R \geq 0$.

CLAIM 7. $d(V_0 \setminus B_R, V_1 \setminus B_R) \rightarrow \infty$ as $R \rightarrow \infty$.

There is a function $F \in C(M^\nu)$ such that $F(V_i) = i$, and let $f = F|_M \in C_\nu(M)$. If Claim 7 were false, there would be some $r > 0$ and sequences $x_{i,k} \in V_i \setminus B_k$ so that $d(x_{0,k}, x_{1,k}) \leq r$ for all k , obtaining the contradiction $1 = f(x_{0,k}) - f(x_{1,k}) \rightarrow 0$ as $k \rightarrow \infty$ because $f \in C_\nu(M)$.

The mapping $R \mapsto d(V_0 \setminus B_R, V_1 \setminus B_R)$ is non-decreasing and upper semi-continuous. It may not be continuous but, using Claim 7, it easily follows that there is a smooth function $\rho : [0, \infty) \rightarrow \mathbb{R}^+$ such that $\rho(R) \leq d(V_0 \setminus B_R, V_1 \setminus B_R)$, $0 \leq \rho' \leq 1$, and $\rho(R) \rightarrow \infty$ as $R \rightarrow \infty$; in particular, $\rho(R) \leq \rho(R+r) \leq \rho(R) + r$ for all $R, r \geq 0$.

Now let $f' : M' \setminus B_1 \rightarrow \mathbb{C}$ be defined by

$$f'(x) = \begin{cases} \frac{\rho(d'(x, x_0)) - d'(x, V_1)}{\rho(d'(x, x_0))} & \text{if } d'(x, V_1) \leq \rho(d'(x, x_0)) \\ 0 & \text{otherwise.} \end{cases}$$

Note that f' is continuous, non-negative and bounded, and $f'(V_i) = i$. Take some $r > 0$ and $x, y \in M'$ such that $d'(x, y) < r$. For $R = d'(x, x_0)$ and $D = d'(x, V_1)$, we have $R - r \leq d'(y, x_0) \leq R + r$ and $D - r \leq d'(y, V_1) \leq D + r$, obtaining

$$\rho(R) - r \leq \rho(R - r) \leq \rho(d'(y, x_0)) \leq \rho(R + r) \leq \rho(R) + r.$$

For the sake of simplicity, let $\rho = \rho(R)$ for this particular R . If $D + 2r \leq \rho$, we get

$$\begin{aligned} f'(x) - f'(y) &= \frac{\rho - D}{\rho} - \frac{\rho - r - (D + r)}{\rho + r} = \frac{\rho - D}{\rho} - \frac{\rho(R) - D + 2r}{\rho + r} \rightarrow 0, \\ f'(y) - f'(x) &\leq \frac{\rho + r - (D - r)}{\rho - r} - \frac{\rho - D}{\rho} = \frac{\rho - D + 2r}{\rho - r} - \frac{\rho - D}{\rho} \rightarrow 0, \end{aligned}$$

as $\rho \rightarrow \infty$ (and therefore as $R \rightarrow \infty$). If $D + 2r \geq \rho$, we get

$$\begin{aligned} f'(x) &= \frac{\rho - D}{\rho} \leq \frac{2r}{\rho} \rightarrow 0, \\ f'(y) &\leq \frac{\rho + r - (D - r)}{\rho - r} = \frac{\rho - D + 2r}{\rho - r} \leq \frac{4r}{\rho - r} \rightarrow 0, \end{aligned}$$

as $\rho \rightarrow \infty$. So $f' \in C_\nu(M')$, and therefore f' has an extension $F' \in C(M'^\nu)$. Also, $f := f'|_M \in C_\nu(M)$, whose continuous extension to M^ν is $F := (\phi^\nu)^* F'$. Since $f(V_i) = i$, we have $F(\text{Cl}_{M^\nu}(V_i)) = i$, and therefore $i = F(e_i) = F'(\phi^\nu(e_i)) = F'(\nu\phi(e_i))$, obtaining $\nu\phi(e_0) \neq \nu\phi(e_1)$. \square

COROLLARY 2.152. *If $\phi : M \rightarrow M'$ is a continuous rough map, then $\phi^\nu : M^\nu \rightarrow M'^\nu$ is an embedding whose image is $\text{Cl}_{M'^\nu}(\phi(M))$.*

PROOF. By Propositions 2.150-(i) and 2.151, ϕ^ν is an injective continuous map between compact Hausdorff spaces. \square

PROPOSITION 2.153 (Álvarez-Candel [ALC11, Proposition 4.12]). *If an open subset $W \subset M$ contains balls of arbitrarily large radius, then*

$$(2.32) \quad \text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(W)) \cap \nu M \neq \emptyset.$$

PROPOSITION 2.154. *For all $W \subset M$ and $r > 0$,*

$$\text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(W)) \cap \nu M = \text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(W_r)) \cap \nu M,$$

where

$$(2.33) \quad W_r = \{x \in W \mid d(x, M \setminus W) > r\}.$$

PROOF. The inclusion “ \supset ” of the statement is obvious.

Let us prove the inclusion “ \subset ” of the statement. For the sake of simplicity, given $W \subset M$ and $r > 0$, let

$$V = \text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(W)), \quad V_r = \text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(W_r)).$$

For any $e \in V \cap \nu M$, there is a continuous function $F : M^\nu \rightarrow [0, 1]$ such that $F(e) = 0$ and $F(M^\nu \setminus V) = 1$. Since $f = F|_M \in C_\nu(M)$, there is a compact $K \subset M$ such that $|V_{r+1}f| < 1/2$ on $M \setminus K$.

CLAIM 8. $f > 1/2$ on $M \setminus (\text{Cl}_{M^\nu}(W_r) \cup K)$.

For all $u \in M \setminus (\text{Cl}_{M^\nu}(W_r) \cup K)$, there is some $v \in M \setminus W$ so that $d(v, w) \leq r < r + 1$. Thus $|f(u) - 1| = |f(u) - f(v)| < 1/2$, obtaining $f(u) > 1/2$, which shows Claim 8.

CLAIM 9. We have

$$M^\nu \setminus (\text{Cl}_{M^\nu}(W_r) \cup K) \subset \text{Cl}_{M^\nu}(M \setminus (\text{Cl}_{M^\nu}(W_r) \cup K)).$$

For any open neighborhood O of a point $e' \in M^\nu \setminus (\text{Cl}_{M^\nu}(W_r) \cup K)$, we get

$$\emptyset \neq (M^\nu \setminus (\text{Cl}_{M^\nu}(W_r) \cup K)) \cap M \cap O = (M \setminus (\text{Cl}_{M^\nu}(W_r) \cup K)) \cap O$$

because $M^\nu \setminus (\text{Cl}_{M^\nu}(W_r) \cup K)$ is open in M^ν , and M is open and dense in M^ν . So $e' \in \text{Cl}_{M^\nu}(M \setminus (\text{Cl}_{M^\nu}(W_r) \cup K))$, showing Claim 9.

From Claims 8 and 9, it follows that $F \geq 1/2$ on $M^\nu \setminus (\text{Cl}_{M^\nu}(W_r) \cup K)$ by the continuity of F . Hence

$$F^{-1}([0, 1/2)) \subset \text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(W_r) \cup K)$$

because $F^{-1}([0, 1/2))$ is open in M^ν . So

$$e \in F^{-1}([0, 1/2)) \cap \nu M \subset \text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(W_r) \cup K) \cap \nu M = V_r \cap \nu M. \quad \square$$

The following corollary states the reciprocal of Proposition 2.153.

COROLLARY 2.155. *If (2.32) holds for some open subset $W \subset M$, then W contains balls of arbitrarily large radius.*

PROOF. By Proposition 2.154, with the notation (2.33), we get $W_r \neq \emptyset$ for all $r > 0$. So $B(x, r) \subset W$ for any $x \in W_r$. \square

COROLLARY 2.156. *If (2.32) holds for some open subset $W \subset M$, then*

$$\text{Cl}_{\nu M}(\text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(W)) \cap \nu M) = \text{Cl}_{M^\nu}(W) \cap \nu M.$$

PROOF. There is a canonical identity $\nu W \equiv \text{Cl}_{M^\nu}(W) \cap \nu M$ by Corollary 2.152. Any open set of νW is of the form $V_0 \cap \nu W$ for some open subset $V_0 \subset W^\nu$. If $V_0 \cap \nu W \neq \emptyset$, then there is some open subset $V_1 \subset W^\nu$ such that $V_1 \cap \nu W \neq \emptyset$ and $\text{Cl}_{M^\nu}(V_1) \subset V_0$. By Corollary 2.155, $V_1 \cap W$ contains balls of arbitrarily large radii by Corollary 2.155. So

$$\emptyset \neq \text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(V_1 \cap W)) \cap \nu M \subset \text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(W)) \cap \nu M$$

by Proposition 2.153. On the other hand,

$$\text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(V_1 \cap W)) \cap \nu M \subset V_0 \cap \text{Cl}_{M^\nu}(W) \cap \nu M = V_0 \cap \nu W.$$

Hence $V_0 \cap \nu W$ meets $\text{Int}_{M^\nu}(\text{Cl}_{M^\nu}(W)) \cap \nu M$. \square

PROPOSITION 2.157. *Let $\phi : M \rightarrow M'$ be an (s_r, c) -rough equivalence. Given a compactification $\overline{M} \leq M^\nu$, according to Lemma 2.148, $\overline{M'} \leq M'^\nu$ be the corresponding compactification, and let $\bar{\phi} : \overline{X} \rightarrow \overline{X'}$ be the map induced by ϕ , whose restriction to the coronas is denoted by $\partial\phi : \partial M \rightarrow \partial M'$. Let $e \in \partial M$, and let \mathcal{V} be a base of neighborhoods of e in \overline{M} . Then*

$$\mathcal{V}' = \{ \text{Cl}_{\overline{M'}}(\text{Pen}_{M'}(\phi(V \cap M), c)) \mid V \in \mathcal{V} \}$$

is a base of neighborhoods of $\partial\phi(e)$ in $\overline{M'}$.

PROOF. Let us consider first the case where $\overline{M} = M^\nu$, and thus $\overline{M'} = M'^\nu$, $\bar{\phi} = \phi^\nu$ and $\partial\phi = \nu\phi$.

Let us prove that any element $\text{Cl}_{M'^\nu}(\text{Pen}_{M'}(\phi(V), c))$ of \mathcal{V}' , defined by some $V \in \mathcal{V}$, is a neighborhood of $e' := \nu\phi(e)$. Let $\psi : M' \rightarrow M$ be another s_r -rough equivalence such that $\psi\phi$ and $\phi\psi$ are c -close to id_M and $\text{id}_{M'}$, respectively. By Proposition 2.150-(i), there is some open neighborhood V' of e' in M'^ν such that $\psi^\nu(V') \subset V$. For any $x \in V' \cap M'$, we have $d'(x, \phi\psi(x)) \leq c$ and $\psi(x) \in V$. So $V' \cap M' \subset \text{Pen}_{M'}(\phi(V), c)$, giving

$$\text{Cl}_{M'^\nu}(V' \cap M') \subset \text{Cl}_{M'^\nu}(\text{Pen}_{M'}(\phi(V), c)).$$

But $V' \subset \text{Cl}_{M'^\nu}(V' \cap M')$ because V' is open in M'^ν and M' is open and dense in M'^ν . So $\text{Cl}_{M'^\nu}(\text{Pen}_{M'}(\phi(V), c))$ is an open neighborhood of e' .

Now, let us prove that any neighborhood V' of e' in M'^ν contains some element of \mathcal{V}' . Take another neighborhood V'_0 of e' in M'^ν with $\text{Cl}_{M'^\nu}(V'_0) \subset V'$. Let $W' = V'_0 \cap M'$ and, for any $r > c$, let W'_r be defined by (2.33). By (2.3) and since $\text{Pen}_{M'}(W'_r, r) \subset W'$, we get $\text{Pen}_{M'}(\text{Cl}_{M'}(W'_r), c) \subset W'$. By Proposition 2.154, the set $V'_{0,r} := \text{Int}_{M'^\nu}(\text{Cl}_{M'^\nu}(W'_r))$ is another neighborhood of e' in M'^ν . By Proposition 2.150-(i), there is some $V \in \mathcal{V}$ such that $\text{Cl}_{M'^\nu}(\phi^\nu(V)) \subset V'_{0,r}$. Hence

$$\begin{aligned} \text{Cl}_{M'^\nu}(\text{Pen}_{M'}(\phi(V \cap M), c)) &\subset \text{Cl}_{M'^\nu}(\text{Pen}_{M'}(V'_{0,r} \cap M', c)) \\ &\subset \text{Cl}_{M'^\nu}(\text{Pen}_{M'}(\text{Cl}_{M'}(W'_r), c)) \subset \text{Cl}_{M'^\nu}(W') \subset \text{Cl}_{M'^\nu}(V'_0) \subset V'. \end{aligned}$$

This completes the proof in the case $\overline{M} = M^\nu$.

Now consider the general case. Let $\pi : M^\nu \rightarrow \overline{M}$ and $\pi' : M'^\nu \rightarrow \overline{M}'$ be continuous extensions of id_M and $\text{id}_{M'}$, respectively. Given e and \mathcal{V} like in the statement, the sets $\tilde{V} := \pi^{-1}(V)$, for $V \in \mathcal{V}$, form a base $\tilde{\mathcal{V}}$ of neighborhoods of any $\tilde{e} \in \pi^{-1}(e)$ in M^ν . So, by the above case, it is easy to see that the sets $\tilde{V}' := \text{Cl}_{M'^\nu}(\text{Pen}_{M'}(\phi(\tilde{V} \cap M), c))$, for $\tilde{V} \in \tilde{\mathcal{V}}$, form a base of neighborhoods of $\nu\phi(\pi^{-1}(e)) = \pi'^{-1}(\partial\phi(e))$ in M'^ν . Since the sets \tilde{V}' are saturated by the fibers of π' , it follows that the sets $\pi'(\tilde{V}') = \text{Cl}_{\overline{M}'}(\text{Pen}_{M'}(\phi(V \cap M), c))$, for $V \in \mathcal{V}$, form a base of neighborhoods of $\partial\phi(e)$. \square

2.9. Asymptotic dimension

Let \mathcal{V} be a cover of a space X . The *multiplicity* of a \mathcal{V} is the least $n \in \mathbb{N} \cup \{\infty\}$ such that there are at most n elements of \mathcal{V} meeting at any point of X . It is said that \mathcal{V} *refines* another cover \mathcal{W} of X if every element of \mathcal{V} is contained in some element of \mathcal{W} . Recall that the *Lebesgue covering dimension* of X is the least $n \in \mathbb{N} \cup \{\infty\}$ such that every open cover of X is refined by a cover with multiplicity $\leq n + 1$.

A family \mathcal{V} of subsets of the metric space M is called *uniformly bounded* if there is some $D > 0$ such that $\text{diam } V \leq D$ for all $V \in \mathcal{V}$. The following definition is somehow dual to the definition of Lebesgue covering dimension.

DEFINITION 2.158. The *asymptotic dimension* of M , denoted²⁶ $\text{asdim } M$, is the least $n \in \mathbb{N} \cup \{\infty\}$ such that any uniformly bounded open cover of X refines some uniformly bounded open cover with multiplicity $\leq n + 1$.

REMARK 2.159. The asymptotic dimension was introduced by Gromov [Gro93] in a different way. We follow the survey [BD11] by Bell and Dranishnikov, which contains many relevant examples and results about the asymptotic dimension.

PROPOSITION 2.160 (See e.g. [BD11, Theorem 1]). *For $n \in \mathbb{N}$, we have $\text{asdim } M \leq n$ if and only if, for any $R > 0$, there exist uniformly bounded families $\mathcal{V}_0, \dots, \mathcal{V}_n$ of subsets of M such that $\bigcup_{i=0}^n \mathcal{V}_i$ is a cover of M , and $d(V, V') > R$ for $V \neq V'$ in any \mathcal{U}_i .*

PROPOSITION 2.161 (See e.g. [BD11, Proposition 2]). *Coarsely equivalent metric spaces have the same asymptotic dimension.*

EXAMPLES 2.162. We have $\text{asdim } T \leq 1$ for any tree T [BD11], $\text{asdim } \mathbb{H}^n = n$ for the hyperbolic space \mathbb{H}^n [Gro93], and $\text{asdim } \mathbb{R}^n = n$ for Euclidean space \mathbb{R}^n [DKU98], [BD11]. For any finitely generated group Γ , we have $\text{asdim } \Gamma = 0$ if and only if Γ is finite [BD11].

²⁶The original notation of Gromov [Gro93] is $\text{asdim}_+ M$.

CHAPTER 3

Pseudogroups

This chapter mainly recalls basic notions and results on pseudogroups, and fixes the notation. Most of these preliminaries can be seen in [Hae85], [Hae88], [Hae02] and [ALC09]. Some new results are also proved.

3.1. Pseudogroups

A collection, \mathcal{H} , of homeomorphisms between open subsets of a topological space Z is called a *pseudogroup* of local transformations of Z (or simply a pseudogroup on Z) if $\text{id}_Z \in \mathcal{H}$, and \mathcal{H} is closed under composition (wherever defined¹), inversion, restriction (to open subsets) and combination (or union) of maps. A subset $E \subset \mathcal{H}$ of the pseudogroup \mathcal{H} is called *symmetric* when $h^{-1} \in E$ for all $h \in E$, and the pseudogroup \mathcal{H} is said to be *generated* by E if every element of \mathcal{H} can be obtained from E by using the above pseudogroup operations. The *restriction* of \mathcal{H} to an open subset U of Z is the pseudogroup on U given by

$$\mathcal{H}|_U = \{ h \in \mathcal{H} \mid \text{dom } h \cup \text{im } h \subset U \}.$$

Let \mathcal{H}' be another pseudogroup on a space Z' . Then $\mathcal{H} \times \mathcal{H}'$ denotes the pseudogroup on $Z \times Z'$ generated by the maps $h \times h'$ with $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$.

A pseudogroup on a space is an obvious generalization of a group acting on a space via homeomorphisms, and so all basic concepts from the theory of group actions can be generalized to pseudogroups. For instance, the *orbit* (or \mathcal{H} -*orbit*) of each $x \in Z$ is the set

$$\mathcal{H}(x) = \{ h(x) \mid h \in \mathcal{H}, x \in \text{dom } h \}.$$

The orbits of \mathcal{H} form a partition of Z . The corresponding quotient space (the orbit space) is denoted by Z/\mathcal{H} .

DEFINITION 3.1 (Haefliger [Hae85], [Hae88]). An *étale morphism* $\Phi : \mathcal{H} \rightarrow \mathcal{H}'$ is a maximal set of homeomorphisms of open subsets of Z to open subsets of Z' such that:

- if $\phi \in \Phi$, $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$, then $h'\phi h \in \Phi$;
- the sources of elements of Φ cover Z ; and,
- if $\phi, \psi \in \Phi$, then $\psi\phi^{-1} \in \mathcal{H}'$.

An étale morphism $\Phi : \mathcal{H} \rightarrow \mathcal{H}'$ is called an *equivalence* if $\Phi^{-1} = \{ \phi^{-1} \mid \phi \in \Phi \}$ is also an étale morphism $\mathcal{H}' \rightarrow \mathcal{H}$, which is called the *inverse* of Φ . An étale morphism $\Phi : \mathcal{H} \rightarrow \mathcal{H}'$ is *generated* by a subset $\Phi_0 \subset \Phi$ if all the elements of Φ can be obtained by combination of composites $h'\phi h$ with $h \in \mathcal{H}$, $\phi \in \Phi_0$ and $h' \in \mathcal{H}'$. The *composite* of two étale morphisms, $\Phi : \mathcal{H} \rightarrow \mathcal{H}'$ and $\Psi : \mathcal{H}' \rightarrow \mathcal{H}''$, is the étale morphism $\Psi\Phi : \mathcal{H} \rightarrow \mathcal{H}''$ generated by $\{ \psi\phi \mid \phi \in \Phi, \psi \in \Psi \}$.

¹Composite of partial maps

An étale morphism $\Phi : \mathcal{H} \rightarrow \mathcal{H}'$ clearly induces a continuous map $\bar{\Phi} : Z/\mathcal{H} \rightarrow Z'/\mathcal{H}'$, which is a homeomorphism if Φ is an equivalence. If \mathcal{H} and \mathcal{H}' are equivalent, then they should be considered as the same generalized dynamical system. Thus the properties of pseudogroups that are invariant by equivalences are especially relevant.

EXAMPLE 3.2. If $U \subset Z$ is an open subset that meets every \mathcal{H} -orbit, then the inclusion map $U \hookrightarrow Z$ generates an equivalence $\mathcal{H}|_U \rightarrow \mathcal{H}$.

REMARK 3.3. Example 3.2 can be used to describe any pseudogroup equivalence $\Phi : \mathcal{H} \rightarrow \mathcal{H}'$ as follows. Let \mathcal{H}'' be the pseudogroup on $Z'' = Z \sqcup Z'$ generated by $\mathcal{H} \cup \mathcal{H}' \cup \Phi$, and let $\Psi : \mathcal{H} \rightarrow \mathcal{H}''$ and $\Psi' : \mathcal{H}' \rightarrow \mathcal{H}''$ be the equivalences generated by $Z \hookrightarrow Z''$ and $Z' \hookrightarrow Z''$, respectively. Then $\Phi = \Psi'^{-1}\Psi$.

The *germ groupoid* of \mathcal{H} is the topological groupoid of germs of maps in \mathcal{H} at all points of their domains, with the operation induced by the composite of partial maps and the étale topology. Its subspace of units can be canonically identified with Z . For each $x \in Z$, the group of elements of this groupoid whose source and range is x will be called the *germ group* of \mathcal{H} at x . The germ groups at points in the same orbit are isomorphic by conjugation in the germ groupoid. Thus the *germ group* of each orbit is defined up to isomorphisms. By Remark 3.3, it follows that, under pseudogroup equivalences, corresponding orbits have isomorphic germ groups.

Let \mathcal{H} be a pseudogroup on a locally compact space Z . Then the orbit space Z/\mathcal{H} is compact if and only if Z has a relatively compact open subset that meets every \mathcal{H} -orbit. The following is a stronger compactness condition on \mathcal{H} .

DEFINITION 3.4 (Haefliger [Hae85]). A pseudogroup, \mathcal{H} , is *compactly generated* if there is a relatively compact open set U in Z , meeting each orbit, such that $\mathcal{H}|_U$ has a finite symmetric set of generators, E , so that each $g \in E$ has an extension $\bar{g} \in \mathcal{H}$ with $\text{dom } \bar{g} \subset \text{dom } g$. In this case, E is called a *system of compact generation* of \mathcal{H} on U .

It was observed in [Hae85] that this notion is invariant by equivalences, and that the relatively compact open set U meeting each orbit can be chosen arbitrarily.

3.2. Coarse quasi-isometry type of orbits

Let \mathcal{H} be a pseudogroup on a space Z , and let E be a symmetric set of generators of \mathcal{H} . Each \mathcal{H} -orbit \mathcal{O} is the set of vertices of a connected graph, defined by attaching an edge to vertices $x, y \in \mathcal{O}$ whenever $y = h(x)$ for some $h \in E$ with $x \in \text{dom } h$. This connected graph structure induces a metric d_E on \mathcal{O} according to Section 2.4.1. For $x \in \mathcal{O}$, $S \subset \mathcal{O}$ and $r \geq 0$, the open and closed r -balls of center x in (\mathcal{O}, d_E) are denoted by $B_E(x, r)$ and $\bar{B}_E(x, r)$, and the r -penumbra of S is denoted by $\text{Pen}_E(S, r)$.

We focus on the following case. Suppose that Z is locally compact and \mathcal{H} is compactly generated. Let $U \subset Z$ be a relatively compact open subset that meets all orbits, let E be a symmetric system of compact generation of \mathcal{H} on U , and let $\mathcal{G} = \mathcal{H}|_U$. Then we consider the metric d_E on the \mathcal{G} -orbits. Even under these conditions, the coarse quasi-isometry type of the \mathcal{G} -orbits may depend on the choice of E [ALC09, Section 6]. In [ALC09], this problem is solved by introducing the following additional condition.

DEFINITION 3.5 (Álvarez-Candel [ALC09, Definition 4.2]). E is called *recurrent* if there exists a relatively compact open subset $V \subset U$ whose intersections with all \mathcal{G} -orbits are equi-nets in the \mathcal{G} -orbits with d_E .

According to the following result, the role played by V in Definition 3.5 can actually be played by any relatively compact open subset of U that meets all \mathcal{G} -orbits.

PROPOSITION 3.6 (Álvarez-Candel [ALC09, Lemma 4.3]). *If E is recurrent and $W \subset U$ is an open subset that meets every \mathcal{G} -orbit, then the intersections of all \mathcal{G} -orbits with W are equi-nets in the \mathcal{G} -orbits with d_E .*

The following result guarantees the existence of recurrent systems of compact generation.

PROPOSITION 3.7 (Álvarez-Candel [ALC09, Corollary 4.5]). *There exists a recurrent system E of compact generation of \mathcal{H} on U such that the extension $\bar{g} \in \mathcal{H}$ of each $g \in E$ with $\text{dom } g \subset \text{dom } \bar{g}$ can be chosen so that $\bar{E} = \{\bar{g} \mid g \in E\}$ is also a recurrent symmetric system of compact generation on some relatively compact open subset $U' \subset Z$ containing \bar{U} .*

Let \mathcal{H}' be another compactly generated pseudogroup on a locally compact space Z' , let U' be a relatively compact open subset of Z' that meets all \mathcal{H}' -orbits, let E' be recurrent symmetric system of compact generation for \mathcal{H}' on U' , and let $\mathcal{G}' = \mathcal{H}'|_{U'}$.

THEOREM 3.8 (Álvarez-Candel [ALC09, Theorem 4.6]). *With the above notation, suppose that there exists an equivalence $\mathcal{H} \rightarrow \mathcal{H}'$, and consider the induced equivalence $\mathcal{G} \rightarrow \mathcal{G}'$ and homeomorphism $U/\mathcal{G} \rightarrow U'/\mathcal{G}'$. Then the \mathcal{G} -orbits, endowed with d_E , are equi-coarsely quasi-isometric to the corresponding \mathcal{G}' -orbits, endowed with $d_{E'}$.*

Theorem 3.8 implies the invariance of the coarse quasi-isometry type of the orbits by equivalences when appropriate representatives of pseudogroups and generators are chosen. The following result gives a more explicit relation between d_E and $d_{E'}$ in a particular case.

PROPOSITION 3.9. *Suppose that $Z = Z'$, $\mathcal{H} = \mathcal{H}'$ and $U \subset U'$. Then d_E and the restrictions of $d_{E'}$ to the \mathcal{G} -orbits are equi-Lipschitz equivalent.*

PROOF. If $U = U'$, this was established in [ALC09, Corollary 4.9]. Assume, by enlarging E if necessary, that $E \subset E'$. Then $d_{E'}(x, y) \leq d_E(x, y)$ for all $x, y \in U$ in the same \mathcal{G} -orbit.

On the other hand, by Proposition 3.6, there is some $R \in \mathbb{N}$ such that $\mathcal{O}' \cap U$ is an R -net in $(\mathcal{O}', d_{E'})$ for every \mathcal{G}' -orbit \mathcal{O}' . Let F be the family composites, f , of at most R maps in E' such that $\text{im } f \subset U$. Then $U' = \bigcup_{f \in F} \text{dom } f$. Let

$$G = \{ f_2^{-1} g' f_1 \mid f_1, f_2 \in F \cup \{\text{id}_U\}, g' \in E' \}.$$

Then G is symmetric because E' is symmetric. Moreover every $g \in G$ has an extension $\bar{g} \in \mathcal{H}$ with $\text{dom } g \subset \text{dom } \bar{g}$ because the elements of E' have such type of extensions. It follows from the definition of G that $E \subset G$ because $E \subset E'$, and therefore G is a recurrent symmetric system of compact generation of \mathcal{H} on U . By [ALC09, Corollary 4.9] (the case $U = U'$), there is $C \geq 1$ such that $d_E(x, y) \leq$

$C d_G(x, y)$ for every $x, y \in U$ in the same \mathcal{G} -orbit. If $x \neq y$ are in the same \mathcal{G} -orbit and have $d_{E'}(x, y) = m \geq 1$, then there are $g'_1, \dots, g'_m \in E'$ such that $y = g'_m \cdots g'_1(x)$, and $f_0, \dots, f_m \in F$ such that $f_1 = f_m = \text{id}_U$ and $g'_k \cdots g'_1(x) \in \text{dom } f_k$ for all $k \in \{1, \dots, m-1\}$. Thus $g_k = f_k g'_k f_{k-1} \in G$ for all $k \in \{1, \dots, m\}$ and $y = g_m \cdots g_1(x)$, obtaining $d_G(x, y) \leq m$. So $d_E(x, y) \leq C d_{E'}(x, y)$. \square

REMARK 3.10. In the case $U = U'$, without assuming that E' is recurrent, [ALC09, Corollary 4.9] in fact states that E' is recurrent if and only if d_E and $d_{E'}$ are equi-Lipschitz equivalent on the \mathcal{G} -orbits.

3.3. A version of local Reeb stability

Let \mathcal{H} be a compactly generated pseudogroup on a locally compact space Z , let U be a relatively compact open subset of Z that meets all \mathcal{H} -orbits, let E be recurrent symmetric system of compact generation of \mathcal{H} on U , and let $\mathcal{G} = \mathcal{H}|_U$.

The following notation will be used. For each $m \in \mathbb{Z}^+$, let E^m denote the m -fold Cartesian product $E \times \cdots \times E$, and set $E^0 = \{\text{id}_Z\}$. For every $g = (g_1, \dots, g_m) \in E^m$, its *domain* is the set $\text{dom } g = \text{dom}(g_1 \cdots g_m)$, which may be empty. Moreover let $g(x) = g_1 \cdots g_m(x)$ for every $x \in \text{dom } g$, and let $g^{-1} = (g_m^{-1}, \dots, g_1^{-1})$. For another $n \in \mathbb{Z}^+$ and $h = (h_1, \dots, h_n) \in E^n$, let $gh = (g_1, \dots, g_m, h_1, \dots, h_n) \in E^{m+n}$. Finally, for $r \in \mathbb{Z}^+$, let $E^{\leq r} = \bigcup_{m=1}^r E^m$.

By Proposition 3.7, each $g \in E$ has an extension \bar{g} such that $\overline{\text{dom } g} \subset \text{dom } \bar{g}$, and the collection $\bar{E} = \{\bar{g} \mid g \in E\}$ is a recurrent symmetric system of compact generation on some relatively compact open subset $U' \subset Z$ with $\bar{U} \subset U'$. Let $\mathcal{G}' = \mathcal{H}|_{U'}$. For $m \in \mathbb{Z}^+$ and $g = (g_1, \dots, g_m) \in E^m$, let $\bar{g} = (\bar{g}_1, \dots, \bar{g}_m) \in \bar{E}^m$. There is some $C \in \mathbb{Z}^+$ such that, for all $x, y \in U$ in the same \mathcal{G} -orbit,

$$(3.1) \quad d_{\bar{E}}(x, y) \leq d_E(x, y) \leq C d_{\bar{E}}(x, y).$$

In (3.1) above, the first inequality holds because \bar{g} is an extension of the corresponding $g \in E$, and the second inequality follows from Proposition 3.9. On the other hand, by Proposition 3.6, there is some $R \in \mathbb{N}$ so that $\mathcal{O}' \cap U$ is an R -net in $(\mathcal{O}', d_{\bar{E}})$ for every \mathcal{G}' -orbit \mathcal{O}' .

Let U_0 be the set of points in U where \mathcal{G} has trivial germ groups. The following is a coarsely quasi-isometric pseudogroup version of the Reeb local stability around points in U_0 , which will play a very important role in the present work.

PROPOSITION 3.11. *For every $r \in \mathbb{Z}^+$ and $x \in U_0$, there exists an open neighborhood $V(x, r)$ of x in U such that:*

- (i) $V(x, r') \subset V(x, r)$ if $r' > r$;
- (ii) $V(x, r) \subset \text{dom } g$ for all $g \in E^{\leq r}$ with $x \in \text{dom } g$;
- (iii) for every $y \in V(x, r)$, a map $\phi_{x,y,r} : \bar{B}_E(x, r) \rightarrow \bar{B}_E(y, r)$ is determined by the condition $\phi_{x,y,r} g(x) = g(y)$ for all $g \in E^{\leq r}$ with $x \in \text{dom } g$;
- (iv) $\phi_{x,y,r}$ is non-expanding with respect to d_E ;
- (v) $\phi_{x,y,r}$ is C -bi-Lipschitz with respect to d_E ; and,
- (vi) if $r \geq CR$, then $\bar{B}_E(y, r/C) \cap \text{im } \phi_{x,y,r}$ is a $2CR$ -net in $(\bar{B}_E(y, r/C), d_E)$.

PROOF. Since all the sets E^m and \bar{E}^m are finite, and since $\overline{\text{dom } g} \subset \text{dom } \bar{g}$ for each $g \in E^m$, there is, for every $r \in \mathbb{Z}^+$ and $x \in U_0$, a largest open neighborhood $V(x, r)$ of x in U satisfying (ii) and the following properties:

- (a) if $g(x) = x$ for some $g \in E^{\leq 4r}$ and $x \in \text{dom } g$, then $V(x, r) \subset \text{dom } g$ and $g(y) = y$ for all $y \in V(x, r)$;
- (b) if $V(x, r)$ meets the domain of some $g \in E^{\leq 4r}$, then $x \in \text{dom } \bar{g}$;
- (c) if $g(y) = y$ for some $g \in E^{\leq 4r}$ and $y \in V(x, r) \cap \text{dom } g$, then $\bar{g}(x) = x$;
- (d) $V(x, r) \subset \text{dom } \bar{g}$ for all $\bar{g} \in \bar{E}^{\leq r}$ with $x \in \text{dom } \bar{g}$; and,
- (e) if $x \in \text{dom } \bar{g}$ and $\bar{g}(x) = x$ for some $\bar{g} \in \bar{E}^{\leq 4r}$, then $V(x, r) \subset \text{dom } \bar{g}$ and $\bar{g}(y) = y$ for all $y \in V(x, r)$.

We also get (i) since $V(x, r)$ is maximal.

Fix $x \in U_0$, $r \in \mathbb{Z}^+$ and $y \in V(x, r)$. Each point in $\bar{B}_E(x, r)$ is of the form $g(x)$ for some $g \in E^{\leq r}$ with $x \in \text{dom } g$, and thus, by (ii), $y \in \text{dom } g$. Suppose that $g(x) = h(x)$ for another $h \in E^{\leq r}$ with $x \in \text{dom } h$. Then $h^{-1}g \in E^{\leq 2r}$, $x \in \text{dom } h^{-1}g$ and $h^{-1}g(x) = x$. By (a), it follows that $y \in \text{dom } h^{-1}g$ and $h^{-1}g(y) = y$; i.e., $g(y) = h(y)$. Therefore the assignment $g(x) \mapsto g(y)$ for $g \in E^{\leq r}$ defines a map $\phi_{x,y,r} : \bar{B}_E(x, r) \rightarrow \bar{B}_E(y, r)$, which is the statement of (iii). For the sake of simplicity, $\phi_{x,y,r}$ will be simply denoted by ϕ in the rest of the proof.

For every $z, z' \in \bar{B}_E(x, r)$ there are $g, g' \in E^{\leq r}$ whose domains contain x and such that $g(x) = z$ and $g'(x) = z'$. If $m = d_E(z, z') \leq 2r$, there is $h \in E^m$ so that $z \in \text{dom } h$ and $h(z') = z$. Thus $x \in \text{dom } g^{-1}hg'$ and $g^{-1}hg'(x) = x$ with $g^{-1}hg' \in E^{\leq 4r}$. It follows from (a) that $y \in \text{dom } g^{-1}hg'$ and $g^{-1}hg'(y) = y$. Hence $g'(y) \in \text{dom } h$ and $hg'(y) = g(y)$, giving $d_E(g(y), g'(y)) \leq m$, which shows (iv).

Let $z, z' \in \bar{B}_E(x, r)$, and let $g, g' \in E^{\leq r}$ be such that $z = g(x)$ and $z' = g'(x)$. Thus $y \in \text{dom } g \cap \text{dom } g'$, $\phi(z) = g(y)$ and $\phi(z') = g'(y)$. If $m = d_E(g(y), g'(y)) \leq 2r$, then there is $h \in E^m$ so that $g'(y) \in \text{dom } h$ and $hg'(y) = g(y)$. Hence $g^{-1}hg' \in E^{\leq 4r}$, $y \in \text{dom } g^{-1}hg'$ and $g^{-1}hg'(y) = y$. By (b) and (c), it follows that $x \in \text{dom } \bar{g}^{-1}\bar{h}\bar{g}'$ and $\bar{g}^{-1}\bar{h}\bar{g}'(x) = x$, and thus $\bar{h}(z) = z'$ with $\bar{h} \in \bar{E}^m$. Therefore

$$d_E(z, z') \leq C d_{\bar{E}}(z, z') \leq C d_E(\phi(z), \phi(z')) ,$$

by (3.1). This shows (v) (considering (iv)).

Observe that (b) and (iv) only require (ii) and (a). Thus, by (d) and (e), using \bar{E} instead of E , for each $y \in V(x, r)$ there is a map $\bar{\phi}_{x,y,r} : \bar{B}_{\bar{E}}(x, r) \rightarrow \bar{B}_{\bar{E}}(y, r)$, which is determined by the condition that $\bar{\phi}_{x,y,r}\bar{g}(x) = \bar{g}(y)$ for all $\bar{g} \in \bar{E}^{\leq r}$ with $x \in \text{dom } \bar{g}$. Moreover $\bar{\phi}_{x,y,r}$ is non-expanding with respect to $d_{\bar{E}}$. Like ϕ , we will simply use the notation $\bar{\phi}$ for $\bar{\phi}_{x,y,r}$. Note that $\bar{\phi} = \phi$ on $\bar{B}_{\bar{E}}(x, r/C) \cap U$, which is contained in $\bar{B}_E(x, r)$ by (3.1). Thus

$$(3.2) \quad \bar{\phi}(\bar{B}_{\bar{E}}(x, r/C) \cap U) \subset \text{im } \phi .$$

On the other hand, for each $z \in \bar{B}_E(y, r/C)$, there is $g \in E^{\leq \lceil r/C \rceil}$ such that $y \in \text{dom } g$ and $z = g(y)$. Hence $x \in \text{dom } \bar{g}$ by (b), $\bar{g}(x) \in \bar{B}_{\bar{E}}(x, r/C)$ because $\bar{g} \in \bar{E}^{\leq \lceil r/C \rceil}$, and $\bar{\phi}\bar{g}(x) = \bar{g}(y) = z$ by (c). So

$$(3.3) \quad \bar{B}_E(y, r/C) \subset \bar{\phi}(\bar{B}_{\bar{E}}(x, r/C)) .$$

Assume that $r \geq CR$. Then $\bar{B}_{\bar{E}}(x, r/C) \cap U$ is a $2R$ -net in $(\bar{B}_{\bar{E}}(x, r/C), d_{\bar{E}})$ by Lemma 2.48. So $\bar{\phi}(\bar{B}_{\bar{E}}(x, r/C) \cap U)$ is a $2R$ -net in $(\bar{\phi}(\bar{B}_{\bar{E}}(x, r/C)), d_{\bar{E}})$ because $\bar{\phi}$ is non-expanding. Hence $\bar{B}_E(y, r/C) \cap \text{im } \phi$ is a $2R$ -net in $(\bar{B}_E(y, r/C), d_{\bar{E}})$ by (3.2) and (3.3), and therefore it is a $2CR$ -net in $(\bar{B}_E(y, r/C), d_E)$ by (3.1), showing (vi). \square

REMARK 3.12. Observe the following in Proposition 3.11:

- (i) According to (i), (ii) and (iii), $\phi_{x,y,r'}|_{\overline{B}_E(x,r)} = \phi_{x,y,r}$ if $r < r'$ and $y \in V(x, r')$.
- (ii) By (ii), (iii) and (iv), it follows that, if $s \geq r > 0$, $x \in U_0$, $y \in V(x, r) \cap U_0$ and $z \in V(y, s) \cap V(x, r)$, then $\text{im } \phi_{x,y,r} \subset \overline{B}_E(y, s)$ and $\phi_{x,z,s} = \phi_{y,z,s} \circ \phi_{x,y,r}$ on $\overline{B}_E(x, r)$.

REMARK 3.13. Let U_E be the complement in U of the \mathcal{G} -saturation of the union of boundaries in U of the domains of the maps in E . Such a U_E is a dense G_δ set, and thus so is $U_0 \cap U_E$. For all $x \in U_0 \cap U_E$, we can choose the open neighbourhoods $V(x, r)$ of Proposition 3.11 satisfying the conditions (ii) and (a) of Proposition 3.11 and its proof, and moreover so that:

- if $V(x, r)$ meets the domain of some $g \in E^{\leq r}$, then $x \in \text{dom } g$; and,
- if $g(y) = y$ for some $g \in E^{\leq 4r}$ and $y \in V(x, r) \cap \text{dom } g$, then $x \in \text{dom } g$ and $g(x) = x$.

Then, arguing like in the proof of Proposition 3.11, it is easy to prove that the maps $\phi_{x,y,r}$ are isometric bijections for all $r > 0$, $x \in U_0 \cap U_E$ and $y \in V(x, r)$.

The following weaker version of Proposition 3.11 is valid for all points of U .

PROPOSITION 3.14. *For every $r \in \mathbb{Z}^+$ and $x \in U$, there exists an open neighborhood $W(x, r)$ of x in U such that:*

- (i) $W(x, r') \subset W(x, r)$ if $r' > r$;
- (ii) $x \in \text{dom } \bar{g}$ for all $g \in E^{\leq r}$ and $y \in W(x, r) \cap \text{dom } g$;
- (iii) for every $y \in W(x, r)$, a map $\xi_{y,x,r} : \overline{B}_E(y, r) \rightarrow \overline{B}_E(x, r)$ is determined by the condition $\xi_{y,x,r} g(y) = \bar{g}(x)$ for all $g \in E^{\leq r}$ with $y \in \text{dom } g$;
- (iv) $\xi_{y,x,r}$ is C -Lipschitz with respect to d_E ; and
- (v) $\overline{B}_E(x, r) \subset \text{im } \xi_{y,x,r}$.

PROOF. Like in the proof of Proposition 3.11, for every $r \in \mathbb{Z}^+$ and $x \in U$, there is a largest open neighborhood $W(x, r)$ of x in U satisfying the following properties:

- (a) $W(x, r) \subset \text{dom } g$ for all $g \in E^{\leq r}$ with $x \in \text{dom } g$;
- (b) $x \in \text{dom } \bar{g}$ for all $g \in E^{\leq 4r}$ and $y \in W(x, r) \cap \text{dom } g$; and
- (c) if $g(y) = y$ for some $g \in E^{\leq 4r}$ and $y \in W(x, r) \cap \text{dom } g$, then $\bar{g}(x) = x$.

Property (i) is also satisfied because $W(x, r)$ is maximal.

Fix $x \in U$, $r \in \mathbb{Z}^+$ and $y \in W(x, r)$. Property (b) is stronger than (ii). Each point in $\overline{B}_E(x, r)$ is of the form $g(x)$ for some $g \in E^{\leq r}$ with $y \in \text{dom } g$, and therefore $x \in \text{dom } \bar{g}$ by (a). Suppose that $g(y) = h(y)$ for another $h \in E^{\leq r}$ with $y \in \text{dom } h$. Then $h^{-1}g \in E^{\leq 2r}$, $y \in \text{dom } h^{-1}g$ and $h^{-1}g(y) = y$. By (c), we get $x \in \text{dom } \overline{h^{-1}g} = \bar{h}^{-1}\bar{g}$ and $\bar{h}^{-1}\bar{g}(x) = x$, obtaining $\bar{g}(x) = \bar{h}(x)$. Therefore a map $\xi_{y,x,r} : \overline{B}_E(y, r) \rightarrow \overline{B}_E(x, r)$ is defined by $g(y) \mapsto \bar{g}(x)$ for $g \in E^{\leq r}$, giving (iii).

For every $z, z' \in \overline{B}_E(y, r)$ there are $g, g' \in E^{\leq r}$ whose domains contain y and such that $g(y) = z$ and $g'(y) = z'$. If $m = d_E(z, z') \leq 2r$, there is $h \in E^m$ so that $z \in \text{dom } h$ and $h(z') = z$. Thus $y \in \text{dom } g^{-1}hg'$ and $g^{-1}hg'(y) = y$ with $g^{-1}hg' \in E^{\leq 4r}$. It follows from (b) and (c) that $x \in \text{dom } \overline{g^{-1}hg'} = \bar{g}^{-1}\bar{h}\bar{g}'$ and $\bar{g}^{-1}\bar{h}\bar{g}'(x) = x$. Hence $\bar{g}'(x) \in \text{dom } \bar{h}$ and $\bar{h}\bar{g}'(x) = \bar{g}(x)$, giving $d_E(\bar{g}(x), \bar{g}'(x)) \leq m$. So

$$d_E(\bar{g}(x), \bar{g}'(x)) \leq C d_E(\bar{g}(x), \bar{g}'(x)) \leq C d_E(z, z')$$

by (3.1), showing (iv).

Let $z \in \overline{B}_E(x, r)$, and let $g \in E^{\leq r}$ be such that $z = g(x)$. Thus $y \in \text{dom } g$ by (a), and $\xi_{y,x,r}g(y) = \bar{g}(x) = g(x)$, obtaining (v). \square

REMARK 3.15. In Proposition 3.14, note the following:

- (i) By (i), (ii) and (iii), $\xi_{y,x,r'}|_{\overline{B}_E(y,r)} = \xi_{y,x,r}$ if $r < r'$ and $y \in W(x, r')$.
- (ii) By (ii), (iii) and Proposition 3.14-(ii),(iii), $\xi_{y,x,r}\phi_{x,y,r} = \text{id}$ on $\overline{B}_E(x, r)$ if $x \in U_0$ and $y \in V(x, r) \cap W(x, r)$.

PROPOSITION 3.16. *For any convergent sequence in U , $x_i \rightarrow x$, and all $r \in \mathbb{Z}^+$,*

$$\overline{B}_E(x, r) \subset \bigcap_i \text{Cl}_U \left(\bigcup_{j \geq i} \overline{B}_E(x_i, r) \right) \subset \overline{B}_E(x, Cr) .$$

PROOF. Like in the proof of Proposition 3.11, for every $r \in \mathbb{Z}^+$ and $x \in U$, there is a largest open neighbourhood P of x in U such that the following properties hold for all $g \in E^{\leq r}$:

- (a) if $x \in \text{dom } g$, then $P \subset \text{dom } g$; and
- (b) if $y \in P \cap \text{dom } g$, then $x \in \text{dom } \bar{g}$.

We can assume that $x_i \in P$ for all i .

The first inclusion of the statement can be proved as follows. Any element of $\overline{B}_E(x, r)$ is of the form $g(x)$ for some $g \in E^{\leq r}$. Then $g(x_i) \in \overline{B}_E(x_i, r)$ for all i by (a), and $g(x_i) \rightarrow g(x)$ as $i \rightarrow \infty$.

Now let us prove the second inclusion. Consider a convergent sequence in U , $z_k \rightarrow z$, such that $z_k \in \overline{B}_E(x_{i_k}, r)$ for indices $i_k \rightarrow \infty$. Thus there are elements $g_k \in E^{\leq r}$ such that $x_{i_k} \in \text{dom } g_k$ and $g_k(x_{i_k}) = z_k$. Since $E^{\leq r}$ is finite, by passing to a subsequence of z_k if needed, we can assume that all maps g_k are equal, and therefore they will be denoted by g . Then $x \in \text{dom } \bar{g}$ by (b), and $z_k = \bar{g}(x_{i_k}) \rightarrow \bar{g}(x)$ as $k \rightarrow \infty$. Thus $z = \bar{g}(x) \in \overline{B}_E(x, r) \subset \overline{B}_E(x, Cr)$ by (3.1). \square

3.4. Topological dynamics

3.4.1. Preliminaries on Baire category. We recall some terminology and results about subsets of a topological space that are relevant to topological dynamics. A good reference for all this and related material is [Kec95].

DEFINITION 3.17. A subset A of a topological space X is called:

- *residual*² if A contains a countable intersection of open dense subsets;
- *nowhere dense* if its closure \overline{A} has empty interior;
- *meager* if A is a countable union of nowhere dense sets (i.e., $X \setminus A$ is residual);
- *Borel* if A is a member of the σ -algebra generated by the open subsets of X ; and
- *Baire*³ if the symmetric difference⁴ $A \Delta U$ is meager for some open $U \subset X$.

The Baire sets of a topological space also form a σ -algebra: the smallest one containing all the open sets and all the meager sets; in particular, every Borel set is a Baire set. A topological space in which every residual subset is dense is called a *Baire space*. Any open subspace of a Baire space is a Baire space. The Baire

²The term *comeager* is also used.

³It is also said that A satisfies the *Baire property*.

⁴Recall that the *symmetric difference* of the sets $A, B \subset X$ is the set $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

category theorem states that every completely metrizable space and every locally compact Hausdorff space is a Baire space [Kec95, Theorem 8.4].

The Kuratowski-Ulam theorem is the topological analog to Fubini's theorem.

THEOREM 3.18 (Kuratowski-Ulam; see e.g. [Kec95, Theorem 8.41]). *Let X and Y be second countable spaces, let $A \subset X \times Y$ be a Baire subset, and let $A_x = \{y \in Y \mid (x, y) \in A\}$ for each $x \in X$. Then the following properties hold:*

- (i) A_x is Baire for residually many $x \in X$.
- (ii) A is meager (respectively, residual) if and only if A_x is meager (respectively, residual) for residually many $x \in X$.

A topological space is called *Polish* if it is separable and completely metrizable; in particular, it is a Baire space. A subspace of a Polish space is Polish if and only if it is a G_δ [Kec95, Theorem 3.11]. A locally compact space is Polish if and only if it is Hausdorff and second countable [Kec95, Theorem 5.3].

3.4.2. Saturated sets. Let \mathcal{H} be a pseudogroup on a space Z . A subset of Z is said to be \mathcal{H} -saturated (or *saturated*) if it is a union of orbits of \mathcal{H} . The *saturation* of a subset $A \subset Z$, denoted by $\mathcal{H}(A)$, is the union of all orbits that meet A ; i.e.,

$$(3.4) \quad \mathcal{H}(A) = \bigcup_h h(A \cap \text{dom } h),$$

where h runs in \mathcal{H} . If a property P is satisfied by the \mathcal{H} -orbits in a residual (respectively, meager) saturated subset of Z , then it will be said that P is satisfied by *residually* (respectively, *meagerly*) many \mathcal{H} -orbits.

LEMMA 3.19. *Let $A \subset B \subset Z$. If A is open, dense or residual in B , then $\mathcal{H}(A)$ is open, dense or residual in $\mathcal{H}(B)$, respectively.*

PROOF. By (3.4), $\mathcal{H}(A)$ is open (respectively, dense) in $\mathcal{H}(B)$ if A is open (respectively, dense) in B . It follows directly from this that $\mathcal{H}(A)$ is residual in $\mathcal{H}(B)$ if A is residual in B . \square

LEMMA 3.20. *If \mathcal{H} is countably generated, then the saturation of a Borel, Baire or meager subset of Z is Borel, Baire or meager, respectively.*

PROOF. Let E be a countable symmetric set of generators of \mathcal{H} , and let S be the countable set of all composites of elements of E , wherever defined. Then the result follows because (3.4) still holds if h runs only in S . \square

REMARK 3.21. Let $R_{\mathcal{H}} \subset Z \times Z$ be the relation set of the equivalence relation “being in the same \mathcal{H} -orbit.” Assume that \mathcal{H} is countably generated. With the notation of the proof of Lemma 3.20, $R_{\mathcal{H}}$ equals the union of the graphs of maps in S , which are easily seen to be F_σ -sets. Hence $R_{\mathcal{H}}$ is an F_σ set because S is countable.

A pseudogroup, \mathcal{H} , is called *transitive* (respectively, *minimal*) if it has a dense orbit (respectively, every orbit is dense). An (\mathcal{H}) -*minimal set* is a non-empty, saturated and closed subset of Z which is minimal among the sets with these properties. The following result is well known.

PROPOSITION 3.22. *If Z is second countable, then the union of all the \mathcal{H} -orbits that are dense is a G_δ -set. In particular, this set is residual if and only if \mathcal{H} is transitive.*

PROOF. Let $\{U_n\}$ be a countable base for the topology of Z . Then the union of all the dense orbits is equal to the intersection of the saturations $\mathcal{H}(U_n)$, which are open in Z . \square

COROLLARY 3.23. *If Z is second countable and \mathcal{H} is transitive, then the union of all the proper minimal sets is meager.*

PROPOSITION 3.24. *If Z is locally compact space and Z/\mathcal{H} is compact, then any nonempty \mathcal{H} -invariant closed subset of Z contains some minimal set.*

PROOF. By the Zorn's lemma, it is enough to prove that any family of \mathcal{H} -invariant nonempty closed subsets $Y_i \subset Z$ has nonempty intersection. By the hypothesis, there is a relatively compact open subset $U \subset Z$ that meets all \mathcal{H} -orbits. Then $\overline{U} \cap Y_i$ is a nonempty compact subset, obtaining $\emptyset \neq \overline{U} \cap \bigcap_i Y_i \subset \bigcap_i Y_i$. \square

The next result is the topological zero-one law, a topological version of ergodicity.

THEOREM 3.25. *If Z is a Baire space and \mathcal{H} is a transitive pseudogroup on Z , then each saturated Baire subset of Z is either meager or residual.*

PROOF. Let A be a saturated Baire subset of Z . There is an open set U such that $A \triangle U$ is a meager set. If A is not meager, then U is non-empty and $U \setminus A$ is meager. Thus, if A is neither meager nor residual, then there are non-empty open subsets U and V so that $A \cap U$ is residual in U and $V \setminus A$ is residual in V . Hence $\mathcal{H}(A \cap U)$ and $\mathcal{H}(V \setminus A)$ are residual in $\mathcal{H}(U)$ and $\mathcal{H}(V)$, respectively (Lemma 3.19). Since there is a dense orbit, the non-empty open sets $\mathcal{H}(U)$ and $\mathcal{H}(V)$ intersect in a non-empty set, and thus $\mathcal{H}(A \cap U)$ meets $\mathcal{H}(V \setminus A)$ because $\mathcal{H}(U)$ and $\mathcal{H}(V)$ are Baire spaces. But $\mathcal{H}(A \cap U) \subset A$ and $\mathcal{H}(V \setminus A) \subset Z \setminus A$, a contradiction. \square

The following theorem is of basic importance for the contents of this work.

THEOREM 3.26 (Hector [Hec77a], Epstein-Millet-Tischler [EMT77]). *If \mathcal{H} is a countably generated pseudogroup on a space Z , then the union of orbits with trivial germ groups is a dense G_δ subset of Z , hence Borel and residual.*

COROLLARY 3.27. *If Z is second countable, and \mathcal{H} is transitive and countably generated, then the union of dense orbits with trivial germ groups is a residual subset of Z .*

PROOF. This is a direct consequence of Proposition 3.22 and Theorem 3.26. \square

3.4.3. The property of being recurrent on Baire sets. The following shows that the condition of being recurrent on systems of compact generation also holds in a Baire sense. It is a topological-coarsely quasi-isometric version of Ghys' "Proposition fondamentale" [Ghy95, p. 402] for pseudogroups.

THEOREM 3.28. *Let \mathcal{H} be a compactly generated minimal pseudogroup of local transformations of a locally compact space Z , let U be a relatively compact open subset of Z that meets all \mathcal{H} -orbits, let E be a recurrent symmetric system of compact generation of \mathcal{H} on U , and let $\mathcal{G} = \mathcal{H}|_U$. Then, for any Baire subset B of U ,*

- (i) *either $\mathcal{G}(B)$ is meager;*
- (ii) *or else the intersections of residually many \mathcal{G} -orbits with B are equi-nets in those \mathcal{G} -orbits with d_E .*

PROOF. Suppose $\mathcal{G}(B)$ is not meager in U . So B is not meager in U by Lemma 3.20. Then there is a non-empty open subset V of U such that $V \setminus B$ is meager in V . Hence $\mathcal{G}(V \setminus B)$ is meager in U by Lemma 3.20, and thus $Y = U \setminus \mathcal{G}(V \setminus B)$ is residual in U and \mathcal{G} -saturated. Since \mathcal{G} is minimal and E recurrent, there is some $R > 0$ such that $\mathcal{O} \cap V$ is an R -net in (\mathcal{O}, d_E) for any \mathcal{G} -orbit \mathcal{O} (Proposition 3.6); but $\mathcal{O} \cap V \subset B$ if $\mathcal{O} \subset Y$, and the result follows. \square

3.4.4. Pseudogroups versus group actions. As said in Section 3.1, a pseudogroup on a space is a generalization of a group acting on a space via homeomorphisms. With a slight change of the topology of the space acted on, the converse is also true, in the following sense.

THEOREM 3.29. *Let \mathcal{H} be a countable generated pseudogroup of local homeomorphisms of a Polish space Z . Then there is a Polish space Z' , with the same underlying set as Z , and a pseudogroup \mathcal{H}' on Z' such that \mathcal{H}' has the same orbits as \mathcal{H} , $\mathcal{H} \subset \mathcal{H}'$, and \mathcal{H}' is equivalent to the pseudogroup generated by a countable group G of homeomorphisms on another Polish space. Moreover, for each symmetric set E of generators of \mathcal{H} , there is a symmetric set F of generators of G , with the same cardinality as E , such that the \mathcal{H} -orbits with d_E are equi-coarsely quasi-isometric to the corresponding G -orbits with d_F .*

PROOF. Let $W = Z \sqcup Z = Z \times \{0, 1\}$. For $i \in \{0, 1\}$, let $\iota_i : Z \rightarrow W$ be given by $\iota_i(z) = (z, i)$, and let $Z_i = \iota_i(Z)$. Let E be a countable symmetric set of generators for \mathcal{H} containing id_Z . For each $h \in E$, let g_h be the Borel measurable bijection of W given by $g_h \iota_0(z) = \iota_1 h(z)$ if $z \in \text{dom } h$, $g_h \iota_1(z) = \iota_0 h^{-1}(z)$ if $z \in \text{im } h$, and $g_h \iota_i(z) = \iota_i(z)$ otherwise; in particular, $g_{\text{id}_Z} \iota_0(z) = \iota_1(z)$ and $g_{\text{id}_Z} \iota_1(z) = \iota_0(z)$ for all $z \in Z$. Then $F = \{g_h \mid h \in E\}$ generates a countable group G of Borel measurable bijections of W . Note that F has the same cardinality as E , and is symmetric because every g_h is of order 2.

By [BK96, Theorem 5.2.1], there is a Polish topology τ^* on W so that G consists of homeomorphisms of the corresponding space W^* , and inducing the same Borel σ -algebra as the original topology of W . Writing $E = \{h_n \mid n \in \mathbb{N}\}$, by [BK96, Theorem 5.1.11], there are Polish topologies τ_n on W such that $\tau^* \subset \tau_0 \subset \tau_1 \subset \dots$, $\iota_0(\text{dom } h_n) \in \tau_n$, and G consists of homeomorphisms of the corresponding spaces. By [BK96, Theorem 5.1.3-(b)], the topology τ' generated by $\bigcup_n \tau_n$ is also Polish. Since $\bigcup_n \tau_n$ is a base of τ' , the maps in G are homeomorphisms of the space W' defined with τ' . Let \mathcal{G} be the pseudogroup generated by G on W' .

Since $Z_0, Z_1 \in \tau'$ and g_{id_Z} restricts to a homeomorphism $Z_0 \rightarrow Z_1$, there is a Polish topology on Z so that the corresponding space Z' satisfies $Z' \sqcup Z' = W'$. Thus the maps $\iota_i : Z' \rightarrow W'$ are open embeddings, the sets Z_i meet all G -orbits, and there is a unique pseudogroup \mathcal{H}' on Z' so that any ι_i generates an equivalence $\mathcal{H}' \rightarrow \mathcal{G}$.

For each $h \in E$, the restriction $g_{\text{id}_Z} g_h : \iota_0(\text{dom } h) \rightarrow \iota_0(\text{im } h)$ is in \mathcal{H}' and corresponds to h via ι_0 . Thus $E \subset \mathcal{H}'$, and therefore $\mathcal{H} \subset \mathcal{H}'$; in particular, the \mathcal{H} -orbits are contained in \mathcal{H}' -orbits.

Suppose that $\mathcal{O} \subset \mathcal{O}'$ for orbits \mathcal{O} of \mathcal{H} and \mathcal{O}' of \mathcal{H}' . For different points, $z \in \mathcal{O}$ and $z' \in \mathcal{O}'$, let $k = d_F(\iota_0(z), \iota_0(z')) \geq 1$, and take $h_1, \dots, h_k \in E$ so that $g_{h_k} \dots g_{h_1} \iota_0(z) = \iota_0(z')$. Then $z \in \text{dom}(h_k \dots h_1)$ and $h_k^{\varepsilon_k} \dots h_1^{\varepsilon_1}(z) = z'$ for some choice of $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$, obtaining that $z' \in \mathcal{O}$ and $d_E(z, z') \leq k$. In particular, this shows that $\mathcal{O} = \mathcal{O}'$. Now let $l = d_E(z, z') \geq 1$, and take $\bar{h}_1, \dots, \bar{h}_l \in E$ such

that $z \in \text{dom}(\bar{h}_l \cdots \bar{h}_1)$ and $\bar{h}_l \cdots \bar{h}_1(z) = z'$. Then, either $g_{\bar{h}_l}^{\delta_l} \cdots g_{\bar{h}_1}^{\delta_1} \iota_0(z) = \iota_0(z')$, or $g_{\text{id}_Z} g_{\bar{h}_l}^{\delta_l} \cdots g_{\bar{h}_1}^{\delta_1} \iota_0(z) = \iota_0(z')$, for some choices of $\delta_1, \dots, \delta_l \in \{\pm 1\}$, obtaining $k \leq l+1$. On the other hand, the intersections of the G -orbits with Z_0 are 1-nets in those G -orbits with d_F because $g_{\text{id}_Z}(Z_1) = Z_0$. So ι_0 defines equi-corse quasi-isometries of the \mathcal{H} -orbits with d_E to the corresponding G -orbits with d_F . \square

REMARK 3.30. (i) The doubling of the space is required because the domain and image of a local homeomorphism $h \in E$ may have non-empty intersection. However, if \mathcal{H} is the representative of the holonomy pseudogroup of a foliated space generated by the transition mappings of a regular foliated atlas (Section 5.1), then that doubling is not necessary because each transition mapping has disjoint image and domain.

- (ii) Theorem 3.29 is related to the result of Feldman and Moore [FM77, Theorem 1] stating the following: *If R countable equivalence relation on a standard Borel space X , then there is a countable group G of Borel automorphisms of X so that R is the orbit equivalence relation induced by G .*

CHAPTER 4

Generic coarse geometry of orbits

The following notation will be used in the whole of this chapter.

HYPOTHESIS 1. A quintuple $(Z, \mathcal{H}, U, \mathcal{G}, E)$ is required to satisfy the following conditions:

- Z is a locally compact Polish space,
- \mathcal{H} is a compactly generated pseudogroup of local transformations of Z ,
- U is a relatively compact open subset of Z that meets all \mathcal{H} -orbits,
- \mathcal{G} denotes the restriction of \mathcal{H} to U , and
- E is a recurrent symmetric system of compact generation of \mathcal{H} on U .

All metric concepts in the \mathcal{G} -orbits will be considered with respect to the metric d_E induced by E . Thus the subindex “ E ” will be deleted from the notation of the metric, open balls, closed balls, and penumbras. Let U_0 denote the union of \mathcal{G} -orbits with trivial germ groups, which is a dense G_δ -subset of U by Theorem 3.26, and therefore U_0 is a Polish subspace (Section 3.4.1). Moreover let $U_{0,d}$ be the union of dense orbits in U_0 , which is a residual subset of Z if \mathcal{H} is transitive (Corollary 3.27). For each $A \subset U_0$, let $\text{Cl}_0(A)$ and $\text{Int}_0(A)$ denote its closure and interior in U_0 , respectively; the same notation will be used for the closure and interior in $U_0 \times U_0$. Assume that $|E| \geq 2$, otherwise the \mathcal{G} -orbits have at most two elements.

4.1. Coarsely quasi-isometric orbits

For $K \in \mathbb{N}$ and $C \in \mathbb{Z}^+$, let $Y(K, C)$ be the set of pairs $(x, y) \in U_0 \times U_0$ such that there is a (K, C) -coarse quasi-isometry $f : A \rightarrow B$ of $\mathcal{G}(x)$ to $\mathcal{G}(y)$ with $x \in A$, $y \in B$ and $f(x) = y$. Notice that $Y(K, C) \subset Y(K', C')$ if $K \leq K'$ and $C \leq C'$.

LEMMA 4.1. *For $K \in \mathbb{N}$ and $C \in \mathbb{Z}^+$, there is some $K' \in \mathbb{N}$ and $C' \in \mathbb{Z}^+$, depending only on K and C , such that $\text{Cl}_0(Y(K, C)) \subset Y(K', C')$.*

PROOF. Let $(x, y) \in \text{Cl}_0(Y(K, C))$. For each $r \in \mathbb{Z}^+$, consider the open neighborhoods $V(x, r)$ and $V(y, r)$ of x and y in U given by Proposition 3.11. Then there is some

$$(x_n, y_n) \in Y(K, C) \cap (V(x, n) \times V(y, Cn))$$

for each $n \in \mathbb{Z}^+$. According to Proposition 3.11, by taking K and C large enough, it can be assumed that all $\phi_n := \phi_{x, x_n, n}$ and $\psi_n := \phi_{y, y_n, Cn}$ are non-expanding equi-bi-Lipschitz maps with bi-Lipschitz distortion C , and the sets $\overline{B}(x_n, n/C) \cap \text{im } \phi_n$ and $\overline{B}(y_n, n) \cap \text{im } \psi_n$ are K -nets in $\overline{B}(x_n, n/C)$ and $\overline{B}(y_n, n)$, respectively. In particular, the restriction

$$\phi_n : \phi_n^{-1}(\overline{B}(x_n, n/C)) \rightarrow \overline{B}(x_n, n/C) \cap \text{im } \phi_n$$

is a (K, C) -coarse quasi-isometry of $\phi_n^{-1}(\overline{B}(x_n, n/C))$ to $\overline{B}(x_n, n/C)$.

On the other hand, for each n , there is a (K, C) -coarse quasi-isometry $f_n : A_n \rightarrow B_n$ of $\mathcal{G}(x_n)$ to $\mathcal{G}(y_n)$, and so that $x_n \in A_n$, $y_n \in B_n$ and $f_n(x_n) = y_n$. Each set $\overline{B}(x_n, n/C) \cap A_n$ is a $2K$ -net in $\overline{B}(x_n, n/C)$; this holds for $\lfloor n/C \rfloor > K$ by Lemma 2.48, and for $\lfloor n/C \rfloor \leq K$ because $x_n \in \overline{B}(x_n, n/C) \cap A_n$. Also, note that

$$f_n(\overline{B}(x_n, n/C) \cap A_n) \subset \overline{B}(y_n, n),$$

and thus each $f_n(\overline{B}(x_n, n/C) \cap A_n)$ is a $2K$ -net in its $2K$ -penumbra P_n in $\overline{B}(y_n, n)$, obtaining that each restriction

$$f_n : \overline{B}(x_n, n/C) \cap A_n \rightarrow f_n(\overline{B}(x_n, n/C) \cap A_n)$$

is a $(2K, C)$ -coarse quasi-isometry of $\overline{B}(x_n, n/C)$ to P_n . Moreover, because each $\overline{B}(y_n, n) \cap \text{im } \psi_n$ is a K -net in $\overline{B}(y_n, n)$, each $P_n \cap \text{im } \psi_n$ is a $2K$ -net in P_n by Lemma 2.48. So the restriction $\psi_n^{-1} : P_n \cap \text{im } \psi_n \rightarrow \psi_n^{-1}(P_n)$ is a $(2K, C)$ -coarse quasi-isometry of P_n to $\psi_n^{-1}(P_n)$. It follows from Proposition 2.12 that, for some $K' \geq 0$ and $C' \geq 1$, depending only on K and C , there is a (K', C') -coarse quasi-isometry g_n of $\phi_n^{-1}(\overline{B}(x_n, n/C))$ to $\psi_n^{-1}(P_n)$; this g_n is a coarse composite of the above three coarse quasi-isometries. Since

$$\begin{aligned} x \in \phi_n^{-1}(\overline{B}(x_n, n/C)) , \quad \phi_n(x) = x_n \in \overline{B}(x_n, n/C) \cap A_n , \\ f_n(x_n) = y_n \in P_n \cap \text{im } \psi_n , \quad \psi_n^{-1}(y_n) = y , \end{aligned}$$

Proposition 2.12 also guarantees that g_n can be chosen so that $x \in \text{dom } g_n$, $y \in \text{im } g_n$ and $g_n(x) = y$.

Observe that

$$\overline{B}(x, n/C) \subset \phi_n^{-1}(\overline{B}(x_n, n/C))$$

because ϕ_n is non-expanding and $\phi_n(x) = x_n$. Therefore the sequence of finite sets $\phi_n^{-1}(\overline{B}(x_n, n/C))$ is exhausting in $\mathcal{G}(x)$. On the other hand,

$$\overline{B}(y_n, n/C^2) \cap B_n \subset f_n(\overline{B}(x_n, n/C) \cap A_n)$$

since $f_n : A_n \rightarrow B_n$ is a C -bi-Lipschitz bijection such that $x_n \in A_n$, $y_n \in B_n$ and $f_n(x_n) = y_n$. Furthermore $\overline{B}(y_n, n/C^2) \cap B_n$ is a $2K$ -net in $\overline{B}(y_n, n/C^2)$, which holds for $\lfloor n/C^2 \rfloor > K$ by Lemma 2.48, and for $\lfloor n/C^2 \rfloor \leq K$ because $y_n \in \overline{B}(y_n, n/C^2) \cap B_n$. Therefore

$$P_n \supset \overline{B}(y_n, n) \cap \text{Pen}(\overline{B}(y_n, n/C^2) \cap B_n, 2K) \supset \overline{B}(y_n, n/C^2) ,$$

giving

$$\overline{B}(y, n/C^2) \subset \psi_n^{-1}(\overline{B}(y_n, n/C^2)) \subset \psi_n^{-1}(P_n)$$

since ψ_n is non-expanding and $\psi_n(y) = y_n$. So the sequence of finite sets $\psi_n^{-1}(P_n)$ is exhausting in $\mathcal{G}(y)$.

By applying Proposition 2.15 to the sequence of coarse quasi-isometries g_n , and using Remark 2.16, it follows that there is a (K', C') -coarse quasi-isometry g of $\mathcal{G}(x)$ to $\mathcal{G}(y)$ such that $x \in \text{dom } g$ and $g(x) = y$; i.e., $(x, y) \in Y(K', C')$. \square

REMARK 4.2. If the statement of Lemma 4.1 were restricted to the residual union of orbits which do not meet the boundaries in U of the domains of maps in E , then it would be an easy consequence of Proposition 2.15 and Remark 3.13.

Let Y be the set of points $(x, y) \in U_0 \times U_0$ such that $\mathcal{G}(x)$ is coarsely quasi-isometric to $\mathcal{G}(y)$.

LEMMA 4.3. $Y = \bigcup_{K, C=1}^{\infty} Y(K, C)$.

PROOF. This equality follows from Corollary 2.33. \square

COROLLARY 4.4. $Y = \bigcup_{K,C=1}^{\infty} \text{Cl}_0(Y(K,C))$; in particular, Y is an F_{σ} -subset of $U_0 \times U_0$.

PROOF. This is elementary by Lemmas 4.1 and 4.3. \square

COROLLARY 4.5. Either $\text{Int}_0(Y(K,C)) \neq \emptyset$ for some $K, C \in \mathbb{Z}^+$, or else Y is a meager subset of $U_0 \times U_0$.

PROOF. If $\text{Int}_0(Y(K,C)) = \emptyset$ for all $K, C \in \mathbb{Z}^+$, then $\text{Int}_0(\text{Cl}_0(Y(K,C))) = \emptyset$ for all $K, C \in \mathbb{Z}^+$ by Lemma 4.1, and thus Y is meager in $U_0 \times U_0$ by Corollary 4.4. \square

THEOREM 4.6. Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. The equivalence relation “ $x \sim y$ if and only if the orbits $\mathcal{G}(x)$ and $\mathcal{G}(y)$ are coarsely quasi-isometric” has a Borel relation set in $U_0 \times U_0$, and has a Baire relation set in $U \times U$; in particular, it has Borel equivalence classes in U_0 , and Baire equivalence classes in U .

PROOF. This follows from Corollary 4.4 and Theorem 3.26. \square

THEOREM 4.7. Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. Suppose that \mathcal{H} is transitive. Then:

- (i) either all \mathcal{G} -orbits in $U_{0,d}$ are equi-coarsely quasi-isometric to each other;
- (ii) or else there exists residually many \mathcal{G} -orbits \mathcal{O} in U_0 such that \mathcal{O} is coarsely quasi-isometric to meagerly many \mathcal{G} -orbits; in particular, there are uncountably many coarse quasi-isometry types of \mathcal{G} -orbits in U_0 in this case.

PROOF. Suppose that $\text{Int}_0(Y(K,C)) \neq \emptyset$ for some $K, C \in \mathbb{Z}^+$. Then all dense $\mathcal{G} \times \mathcal{G}$ -orbits in $U_0 \times U_0$ meet $Y(K,C)$, and thus all \mathcal{G} -orbits in $U_{0,d}$ are equi-coarsely quasi-isometric to each other.

If $\text{Int}_0(Y(K,C)) = \emptyset$ for all $K, C \in \mathbb{Z}^+$, then Y is meager in $U_0 \times U_0$ by Corollary 4.5. It follows from Theorem 3.18 that there is a residual subset $A \subset U_0$ such that $Y_x = \{y \in U_0 \mid (x,y) \in Y\}$ is meager in U for all $x \in A$. But each Y_x is the union of orbits in U_0 which are coarsely quasi-isometric to $\mathcal{G}(x)$. Finally, A can be assumed to be \mathcal{G} -saturated since Y is $\mathcal{G} \times \mathcal{G}$ -saturated. \square

THEOREM 4.8. Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. Then the following properties hold:

- (i) Suppose that \mathcal{H} is transitive. If there is a coarsely quasi-symmetric \mathcal{G} -orbit in $U_{0,d}$, then the alternative (i) of Theorem 4.7 holds.
- (ii) Suppose that \mathcal{H} is minimal. If the alternative (i) of Theorem 4.7 holds, then all \mathcal{G} -orbits in U_0 are equi-coarsely quasi-symmetric.

PROOF. Suppose that \mathcal{H} is transitive and that there is a coarsely quasi-symmetric \mathcal{G} -orbit \mathcal{O} in $U_{0,d}$. Then $\mathcal{O} \times \mathcal{O} \subset Y(K,C)$ for some $K, C \in \mathbb{Z}^+$, giving $U_0 \times U_0 = Y(K',C')$ for $K', C' \in \mathbb{Z}^+$ by Lemma 4.1, which means that all \mathcal{G} -orbits in U_0 are equi-coarsely quasi-isometric to each other.

Assume that \mathcal{H} is minimal and that all \mathcal{G} -orbits in U_0 are coarsely quasi-isometric to each other. This means that $Y = U_0 \times U_0$. Hence $\text{Int}_0(Y(K,C)) \neq \emptyset$ for some $K, C \in \mathbb{Z}^+$ by Corollary 4.5; i.e., there are some non-empty open subsets V and W of U_0 so that $V \times W \subset Y(K,C)$. But, by Proposition 3.6 and since \mathcal{G} is minimal, the intersections $\mathcal{O} \cap V$ and $\mathcal{O} \cap W$ are equi-nets in the \mathcal{G} -orbits \mathcal{O} in U_0 . So the \mathcal{G} -orbits in U_0 are equi-coarsely quasi-symmetric by Lemma 2.62. \square

The following result follows from Theorems 4.8 and 2.146.

COROLLARY 4.9. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. Suppose that \mathcal{G} is minimal and satisfies the alternative (i) of Theorem 4.7. Then all \mathcal{G} -orbits in U_0 have zero, one, two or a Cantor space of coarse ends, simultaneously.*

4.2. Growth of the orbits

4.2.1. Orbits with the same growth type. Since the \mathcal{G} -orbits are equi-quasi-lattices in themselves (Example 2.51-(i)), the growth type of $\mathcal{G}(x)$ is represented by the mapping $r \mapsto v(x, r) = |\overline{B}(x, r)|$ for all $x \in U$. For $a, b \in \mathbb{Z}^+$ and $c \in \mathbb{N}$, let

$$Y(a, b, c) = \{ (x, y) \in U_0 \times U_0 \mid v(x, r) \leq a v(y, br) \ \forall r \geq c \}.$$

Note that

$$a \leq a', \ b \leq b', \ c \leq c' \implies Y(a, b, c) \subset Y(a', b', c').$$

LEMMA 4.10. *For all $a, b \in \mathbb{Z}^+$ and $c \in \mathbb{N}$, there are some integers $a' \geq a$ and $b' \geq b$ such that $\text{Cl}_0(Y(a, b, c)) \subset Y(a', b', c)$.*

PROOF. Consider the notation of Proposition 3.11. Let $(x, y) \in \text{Cl}_0(Y(a, b, c))$. For any integer $r \geq c$, take a pair

$$(x', y') \in Y(a, b, c) \cap (V(x, r) \times V(y, Cbr)).$$

Then, with the notation given by (2.12),

$$v(x, r) \leq v(x', r) \leq a v(y', br) \leq a \Lambda_{|E|, 2RC} v(y, Cbr)$$

since $\phi_{x, x', r}$ is injective (Proposition 3.11-(v)) and $\overline{B}(y', br) \cap \text{im } \phi_{y, y', Cbr}$ is a $2RC$ -net in $\overline{B}(y', br)$ (Proposition 3.11-(vi)), and because $|\overline{B}(z, 2RC)| \leq \Lambda_{|E|, 2RC}$ for all $z \in \overline{B}(y', br)$ by (2.13). Hence $(x, y) \in Y(a \Lambda_{|E|, 2RC}, Cb, c)$. \square

Note that $Y = \bigcup_{a, b, c=1}^{\infty} Y(a, b, c)$ is the set of points $(x, y) \in U_0 \times U_0$ such that the growth type of $\mathcal{G}(x)$ is dominated by the growth type of $\mathcal{G}(y)$. Let $\tau : U \times U \rightarrow U \times U$ be the homeomorphism given by $\tau(x, y) = (y, x)$

THEOREM 4.11. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. The equivalence relation “ $x \sim y$ if and only if the orbits $\mathcal{G}(x)$ and $\mathcal{G}(y)$ have the same growth type” has a Borel relation set in $U_0 \times U_0$, and has a Baire relation set in $U \times U$; in particular, it has Borel equivalence classes in U_0 , and Baire equivalence classes in U .*

PROOF. By Lemma 4.10, Y is an F_σ -subset of $U_0 \times U_0$, and thus so is $Y_\tau := Y \cap \tau(Y)$. But Y_τ is the relation set in $U_0 \times U_0$ of the statement, and the result follows because U_0 is residual in U (Theorem 3.26). \square

THEOREM 4.12. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. If \mathcal{G} is transitive, then:*

- (i) *either all \mathcal{G} -orbits in $U_{0,d}$ have equi-equivalent growth;*
- (ii) *or else there exist residually many \mathcal{G} -orbits \mathcal{O} in U_0 such that the growth type of \mathcal{O} is comparable with the growth type of meagerly many \mathcal{G} -orbits; in particular, there are uncountably many growth types of \mathcal{G} -orbits in U_0 in this second case.*

PROOF. If $\text{Int}_0(Y(a, b, c)) \neq \emptyset$ for some $a, b, c \in \mathbb{Z}^+$, then all dense $\mathcal{G} \times \mathcal{G}$ -orbits in $U_0 \times U_0$ meet $Y(a, b, c)$. It follows that all \mathcal{G} -orbits in $U_{0,d}$ have equi-equivalent growth.

On the other hand, if $\text{Int}_0(Y(a, b, c)) = \emptyset$ for all $a, b, c \in \mathbb{Z}^+$, then Y is a meager subset of $U_0 \times U_0$ by Lemma 4.10, and thus it is meager in $U \times U$ too. So $Y^\tau := Y \cup \tau(Y)$ is meager in $U \times U$ as well. It follows from Theorem 3.18 that there is a residual subset $A \subset U_0$ such that $Y_x^\tau = \{y \in U_0 \mid (x, y) \in Y^\tau\}$ is meager for all $x \in A$. But each Y_x^τ is the union of \mathcal{G} -orbits in U_0 whose growth type is comparable with the growth type of $\mathcal{G}(x)$. Obviously, it can be assumed that A is saturated. \square

THEOREM 4.13. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. Then the following properties hold:*

- (i) *Suppose that \mathcal{H} is transitive. If there is a growth symmetric \mathcal{G} -orbit in $U_{0,d}$, then the alternative (i) of Theorem 4.12 holds.*
- (ii) *Suppose that \mathcal{H} is minimal. If the alternative (i) of Theorem 4.12 holds, then all \mathcal{G} -orbits in U_0 are equi-growth symmetric.*

PROOF. Suppose that \mathcal{G} is transitive and that $\mathcal{G}(x)$ is growth symmetric for some $x \in U_{0,d}$. Then $\mathcal{G}(x) \times \mathcal{G}(x) \subset Y(a, b, c)$ for some $a, b \geq 1$ and $c \geq 0$, giving $U_0 \times U_0 = Y(a', b', c)$ for some $a', b' \geq 1$ by Lemma 4.10.

Now, assume that \mathcal{G} is minimal and that all \mathcal{G} -orbits in U_0 have equi-equivalent growth. This means that $\text{Int}_0(Y(a, b, c)) \neq \emptyset$ for some $a, b, c \in \mathbb{Z}^+$ according to the proof of Theorem 4.12; i.e., there are non-empty open subsets V and W of U_0 such that $V \times W \subset Y(a, b, c)$. Since \mathcal{G} is minimal, the intersections $\mathcal{O} \cap V$ and $\mathcal{O} \cap W$ are equi-nets in the \mathcal{G} -orbits \mathcal{O} in U_0 by Proposition 3.6. So the \mathcal{G} -orbits in U_0 is equi-growth symmetric by Remark 2.87-(ii). \square

4.2.2. Some growth classes of the orbits.

THEOREM 4.14. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1 and suppose that \mathcal{G} is transitive. Then there are $a_1, a_3 \in [1, \infty]$, $a_2, a_4 \in [0, \infty)$ and $p \geq 1$ such that*

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log v(x, r)}{\log r} &= a_1, & a_2 &\leq \liminf_{r \rightarrow \infty} \frac{\log v(x, r)}{r} \leq pa_2, \\ \liminf_{r \rightarrow \infty} \frac{\log v(x, r)}{\log r} &= a_3, & \limsup_{r \rightarrow \infty} \frac{\log v(x, r)}{r} &= a_4 \end{aligned}$$

for residually many points x in U . Moreover

$$\liminf_{r \rightarrow \infty} \frac{\log v(x, r)}{\log r} \geq a_3, \quad \limsup_{r \rightarrow \infty} \frac{\log v(x, r)}{r} \leq a_4$$

for all $x \in U_{0,d}$.

PROOF. For $a, b > 0$, let $Y_1(a, b)$, $Y_2(a, b)$, $Y_3(a, b)$ and $Y_4(a, b)$ be the sets of points $x \in U_0$ that satisfy the following respective conditions:

$$\begin{aligned} \sup_{r \geq b} \frac{\log v(x, r)}{\log r} &\leq a, & \inf_{r \geq b} \frac{\log v(x, r)}{r} &\geq a, \\ \inf_{r \geq b} \frac{\log v(x, r)}{\log r} &< a, & \sup_{r \geq b} \frac{\log v(x, r)}{r} &> a. \end{aligned}$$

Then

$$(4.1) \quad a \leq a' \ \& \ b \leq b' \implies \begin{cases} Y_1(a, b) \subset Y_1(a', b'), & Y_2(a', b) \subset Y_2(a, b'), \\ Y_3(a, b') \subset Y_3(a', b), & Y_4(a', b') \subset Y_4(a, b), \end{cases}$$

CLAIM 10. The following properties hold:

- (i) $Y_1(a, b)$ is closed in U_0 for all $a, b > 0$.
- (ii) There is some $p \geq 1$ such that, for all $a, b > 0$, there is some $b' \geq b$ so that $\text{Cl}_0(Y_2(a, b)) \subset Y_2(a/p, b')$.
- (iii) $\cap_b Y_3(a, b) \subset \cap_{a' > a} \text{Int}_0(\cap_b Y_3(a', b))$ for all $a > 0$.
- (iv) $Y_4(a, b)$ is open in U_0 for all $a, b > 0$.

Let $x \in \text{Cl}_0(Y_1(a, b))$. With the notation of Proposition 3.11, for each $r \geq b$, if $y \in Y_1(a, b) \cap V(x, r)$, then

$$\frac{\log v(x, r)}{\log r} = \frac{\log v(x, \lfloor r \rfloor)}{\log r} \leq \frac{\log v(y, \lfloor r \rfloor)}{\log r} = \frac{\log v(y, r)}{\log r} \leq a$$

because $\phi_{x, y, \lfloor r \rfloor}$ is injective by Proposition 3.11-(v). Therefore $x \in Y_1(a, b)$, confirming Claim 10-(i).

Let $x \in \text{Cl}_0(Y_2(a, b))$, and let $\Lambda := \Lambda_{|E|, 2RC}$ be defined by (2.12). If

$$r \geq b' := \max \left\{ Cb, CR, \frac{2C \log \Lambda}{a} \right\}$$

and $y \in Y_2(a, b) \cap V(x, r)$, then

$$\begin{aligned} \frac{\log v(x, r)}{r} &= \frac{\log v(x, \lfloor r \rfloor)}{r} \geq \frac{\log(v(y, \lfloor r \rfloor)/C)/\Lambda}{r} = \frac{\log v(y, \lfloor r \rfloor/C) - \log \Lambda}{r} \\ &= \frac{\log v(y, r/C) - \log \Lambda}{r} \geq \frac{\log v(y, r/C)}{r} - \frac{a}{2C} \geq \frac{a}{C} - \frac{a}{2C} = \frac{a}{2C}, \end{aligned}$$

using that $\overline{B}(y, \lfloor r \rfloor/C) \cap \text{im } \phi_{x, y, \lfloor r \rfloor}$ is a $2RC$ -net in $\overline{B}(y, \lfloor r \rfloor/C)$ according to Proposition 3.11-(vi), and $|\overline{B}(z, 2RC)| \leq \Lambda$ for all $z \in \overline{B}(y, r/C)$ by (2.13). Thus $x \in Y_2(a/p, b')$ for $p = 2C$, which confirms Claim 10-(ii).

Let $x \in \cap_b Y_3(a, b)$. Given $a' > a$, for any choice of $\alpha \in (a/a', 1)$, let

$$b \geq \max \left\{ CR, C^{\frac{1}{1-\alpha}}, C\Lambda^{\frac{1}{\alpha'-a/\alpha}} \right\},$$

where Λ is defined like in the proof of Claim 10-(ii). Then $x \in Y_3(a, b)$, which means that there is some $r \geq b$ so that $\log v(x, r)/\log r < a$. If $y \in V(x, Cr)$, then

$$\begin{aligned} \frac{\log v(y, r/C)}{\log(r/C)} &= \frac{\log v(y, \lfloor r \rfloor/C)}{\log(r/C)} \leq \frac{\log(v(x, \lfloor r \rfloor)\Lambda)}{\log(r/C)} = \frac{\log(v(x, r)\Lambda)}{\log(r/C)} \\ &\leq \frac{\log v(x, r) + \log \Lambda}{\log r - \log C} < \frac{a}{1 - \frac{\log C}{\log r}} + \frac{\log \Lambda}{\log r - \log C} < \frac{a}{\alpha} + a' - \frac{a}{\alpha} = a', \end{aligned}$$

like in the proof of Claim 10-(ii). It follows that $V(x, Cr) \cap U_0 \subset Y_3(a', b)$. Since this holds for all b large enough, we get $V(x, Cr) \cap U_0 \subset \cap_b Y_3(a', b)$ by (4.1), and thus $x \in \text{Int}_0(\cap_b Y_3(a', b))$, showing Claim 10-(iii).

For any $x \in Y_4(a, b)$, there is some integer $r \geq b$ such that $\log v(x, r)/r > a$. So $\log v(y, r)/r > a$ for any $y \in V(x, r)$ since $\phi_{x, y, r}$ is injective (Proposition 3.11-(v)), giving $V(x, r) \cap U_0 \subset Y_4(a, b)$. Therefore $Y_4(a, b)$ is open in U_0 . This confirms Claim 10-(iv).

For $a \in [0, \infty]$, let

$$\begin{aligned} Y_1(a) &= \bigcap_{\alpha > a} \bigcup_b Y_1(\alpha, b), & Y_2(a) &= \bigcap_{\alpha < a} \bigcup_b Y_2(\alpha, b), \\ Y_3(a) &= \bigcap_{\alpha > a} \bigcap_b Y_3(\alpha, b), & Y_4(a) &= \bigcap_{\alpha < a} \bigcap_b Y_4(\alpha, b). \end{aligned}$$

It is clear that these are the sets of points $x \in U_0$ that respectively satisfy

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log v(x, r)}{\log r} &\leq a, & \liminf_{r \rightarrow \infty} \frac{\log v(x, r)}{r} &\geq a, \\ \liminf_{r \rightarrow \infty} \frac{\log v(x, r)}{\log r} &\leq a, & \limsup_{r \rightarrow \infty} \frac{\log v(x, r)}{r} &\geq a. \end{aligned}$$

Observe also that

$$(4.2) \quad a \leq a' \implies \begin{cases} Y_1(a) \subset Y_1(a'), & Y_2(a) \supset Y_2(a'), \\ Y_3(a) \subset Y_3(a'), & Y_4(a) \supset Y_4(a'). \end{cases}$$

We get the same sets $Y_1(a)$, $Y_2(a)$, $Y_3(a)$ and $Y_4(a)$ above if the condition that $a', b \in \mathbb{Q}$ is added in their definitions. So, by Claim 10-(i),(iii),(iv), the sets $Y_1(a)$, $Y_3(a)$ and $Y_4(a)$ are Borel in U_0 , and therefore they are Baire subsets of U . However the same kind of argument, using Claim 10-(ii), does not apply to $Y_2(a)$. Thus consider also the set

$$Y_2'(a) = \bigcap_{\alpha < a} \bigcup_b \text{Cl}_0(Y_2(\alpha, b)),$$

which are Borel in U_0 and Baire in U . Obviously,

$$(4.3) \quad a \leq a' \implies Y_2'(a) \supset Y_2'(a'),$$

and, by Claim 10-(ii),

$$(4.4) \quad Y_2'(pa) \subset Y_2(a).$$

The sets $Y_1(a)$ and $Y_3(a)$ are \mathcal{G} -saturated for all $a \in [0, \infty]$ by Remark 2.78-(ii), and we have

$$(4.5) \quad \mathcal{G}(Y_2(qa)) \subset Y_2(a), \quad \mathcal{G}(Y_4(qa)) \subset Y_4(a)$$

for all $a \in [0, \infty]$ and $q > 1$ by Remark 2.78-(iii) and Example 2.51-(i) since $\Lambda_{|E|,0} = 1$ (see (2.12)). So, by (4.4),

$$(4.6) \quad \mathcal{G}(Y_2'(pqa)) \subset Y_2(a).$$

CLAIM 11. Each of the sets $Y_1(a)$, $\mathcal{G}(Y_2'(a))$, $Y_3(a)$ and $\mathcal{G}(Y_4(a))$ is either residual or meager in U .

This assertion is a consequence of Theorem 3.25 because the stated sets are Baire and \mathcal{G} -saturated in U .

We obviously have

$$(4.7) \quad Y_1(\infty) = Y_2(0) = Y_3(\infty) = Y_4(0) = U_0.$$

Then let

$$\begin{aligned} a_1 &= \inf \{ a \in [0, \infty] \mid Y_1(a) \text{ is residual in } U \}, \\ a_2 &= \sup \{ a \in [0, \infty] \mid Y_2(a) \text{ is residual in } U \}, \\ a_3 &= \inf \{ a \in [0, \infty] \mid Y_3(a) \text{ is residual in } U \}, \\ a_4 &= \sup \{ a \in [0, \infty] \mid Y_4(a) \text{ is residual in } U \}. \end{aligned}$$

Since the growth type of an infinite connected graph of finite type is at least linear and at most exponential, the sets $Y_1(a)$ and $Y_3(a)$ are the union of finite orbits in U_0 for all $0 \leq a < 1$, and $Y_2(\infty) = Y_4(\infty) = \emptyset$. So $a_1, a_3 \geq 1$ by Corollary 3.27, and $a_2, a_4 < \infty$. By (4.2), the sets $Y_1(a)$, $Y_2(a)$, $Y_3(a)$ or $Y_4(a)$ are residual in U if $a > a_1$, $a < a_2$, $a > a_3$ or $a < a_4$, respectively. Hence, by (4.7), and because

$$\begin{aligned} Y_1(a_1) &= \bigcap_{a_1 < a < \infty} Y_1(a) \quad \text{if } a_1 < \infty, \\ Y_2(a_2) &= \bigcap_{0 < a < a_2} Y_2(a) \quad \text{if } a_2 > 0, \\ Y_3(a_3) &= \bigcap_{a_1 < a < \infty} Y_3(a) \quad \text{if } a_3 < \infty, \\ Y_4(a_4) &= \bigcap_{0 < a < a_4} Y_4(a) \quad \text{if } a_4 > 0, \end{aligned}$$

where a can be taken in \mathbb{Q} , we get that $Y_1(a_1)$, $Y_2(a_2)$, $Y_3(a_3)$ and $Y_4(a_4)$ are residual in U . On the other hand, by (4.2), (4.3), (4.5), (4.6) and Claim 11, the sets $Y_1(a)$, $Y_2(a)$, $Y_3(a)$ or $Y_4(a)$ are meager in U if $a < a_1$, $a > pa_2$, $a < a_3$ or $a > a_4$, respectively. So the following unions are meager in U because they do not change if a is taken in \mathbb{Q} :

$$\bigcup_{0 \leq a < a_1} Y_1(a), \quad \bigcup_{pa_2 < a < \infty} Y_2(a), \quad \bigcup_{0 \leq a < a_3} Y_3(a), \quad \bigcup_{a_4 < a < \infty} Y_4(a).$$

Thus the following sets are residual in U :

$$\begin{aligned} Y_1(a_1) \setminus \bigcup_{0 \leq a < a_1} Y_1(a), \quad Y_2(a_2) \setminus \bigcup_{pa_2 < a < \infty} Y_2(a), \\ Y_3(a_3) \setminus \bigcup_{0 \leq a < a_3/p'} Y_3(a), \quad Y_4(a_4) \setminus \bigcup_{a_4 < a < \infty} Y_4(a). \end{aligned}$$

These are the sets described by the first group of equalities and inequalities of the statement.

Now, let us prove the last two inequalities of the statement on $U_{0,d}$. By Claim 10-(iii),

$$Y_3(a) = \bigcap_{a' > a} \text{Int}_0\left(\bigcap_b Y_3(a, b)\right)$$

for all $a \in [0, \infty]$. So

$$U_0 \setminus Y_3(a) = U_0 \setminus \bigcap_{a' > a} \text{Int}_0\left(\bigcap_b Y_3(a, b)\right) = \bigcup_{a' > a} (U_0 \setminus \text{Int}_0(\bigcap_b Y_3(a, b))).$$

Here, we can take a' in \mathbb{Q} , and thus this expression is a countable union of closed subsets of U_0 . Moreover we know that $U_0 \setminus Y_3(a)$ is residual in U_0 for $a < a_3$, and therefore there is some b such that $U_0 \setminus \text{Int}_0(\bigcap_b Y_3(a, b))$ has nonempty interior, obtaining that any \mathcal{G} -orbit in $U_{0,d}$ meets $U_0 \setminus Y_3(a)$. Hence $U_{0,d} \subset U_0 \setminus Y_3(a)$ for all $a < a_3$ because $Y_3(a)$ is \mathcal{G} -saturated, obtaining that $U_{0,d} \subset U_0 \setminus \bigcup_{a < a_3} Y_3(a)$.

Finally,

$$U_0 \setminus Y_4(a) = U_0 \setminus \bigcap_{a' < a} \bigcap_b Y_4(a', b) = \bigcup_{a' < a} \bigcup_b (U_0 \setminus Y_4(a', b)),$$

where a' and b can be taken in \mathbb{Q} . Then, by Claim 10-(iv), this expression is a countable union of closed subsets of U_0 . Furthermore $U_0 \setminus Y_4(a)$ is residual for each $a > a_4$, obtaining that $U_0 \setminus Y_4(a', b)$ has nonempty interior in U_0 for some $a' < a$ and b . So any \mathcal{G} -orbit in $U_{0,d}$ meets $U_0 \setminus Y_4(a)$ for all $a > a_4$, and therefore

$$U_{0,d} \subset \mathcal{G}(U_0 \setminus Y_4(a)) \subset U_0 \setminus \mathcal{G}(Y_4(a')) \subset U_0 \setminus Y_4(a')$$

if $a' > a > a_4$ by (4.5). Thus $U_{0,d} \subset U_0 \setminus \bigcup_{a>a_4} Y_4(a)$. \square

COROLLARY 4.15. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1 and suppose that \mathcal{G} is transitive. Then each of the following sets is either meager or residual in U :*

- (i) *the union of \mathcal{G} -orbits in U_0 with polynomial growth;*
- (ii) *the union of \mathcal{G} -orbits in U_0 with exponential growth;*
- (iii) *the union of \mathcal{G} -orbits in U_0 with quasi-polynomial growth;*
- (iv) *the union of \mathcal{G} -orbits in U_0 with quasi-exponential growth; and*
- (v) *the union of \mathcal{G} -orbits in U_0 with pseudo-quasi-polynomial growth.*

Moreover,

- (a) *if the set (iii) is residual in U , then it contains $X_{0,d}$; and,*
- (b) *if one of the sets (iv) or (v) is meager in U , then it does not meet $U_{0,d}$.*

REMARK 4.16. The properties used in Corollary 4.15 depend only on the growth type of the \mathcal{G} -orbits by Remark 2.78-(i).

4.3. Amenable orbits

Like in the above section, since the \mathcal{G} -orbits are equi-quasi-lattices in themselves, that they are (equi-) amenable means that they are (equi-) Følner.

THEOREM 4.17. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. Then the following properties hold:*

- (i) *If \mathcal{G} is transitive and some \mathcal{G} -orbit in U_0 is amenable, then all \mathcal{G} -orbits in $U_{0,d}$ are equi-amenable.*
- (ii) *If \mathcal{G} is minimal and some \mathcal{G} -orbit in U_0 is amenable, then all \mathcal{G} -orbits in U_0 are jointly amenable symmetric.*

PROOF. Consider the notation of Proposition 3.11. Suppose that $\mathcal{G}(x)$ is Følner for some $x \in U_0$. Let S_n be a Følner sequence for $\mathcal{G}(x)$, and let $r \in \mathbb{N}$. For each n , let $u_n \geq C(r + 4CR)$ be an integer such that

$$(4.8) \quad S_n \subset \overline{B}(x, u_n/C - r - 4CR).$$

For $y \in V(x, u_n)$, let ϕ_n denote ϕ_{x,y,u_n} , and let $S_{y,n} = \text{Pen}(\phi_n(S_n), 2CR)$. By (4.8) and Proposition 3.11-(iv),

$$(4.9) \quad S_{y,n} \subset \overline{B}(y, u_n/C - r - 2CR).$$

Moreover, by Proposition 3.11-(v),

$$(4.10) \quad |S_n| \leq |S_{y,n}|.$$

CLAIM 12. $\partial_r S_{y,n} \subset \text{Pen}(\phi_n(\partial_{C(r+4CR)} S_n), 2CR)$.

Observe that the right hand side of this inclusion is well defined by (4.8). To prove this inclusion, take first any $z \in \partial_r S_{y,n} \setminus S_{y,n}$. Then $d(z, \phi_n(S_n)) > 2CR$ and there is some $z_0 \in S_{n,y}$ such that $d(z, z_0) \leq r$. Thus there is some $z'_0 \in \phi_n(S_n)$ so that $d(z_0, z'_0) \leq 2CR$, obtaining $d(z, z'_0) \leq 2CR + r$ by the triangle inequality. By (4.9),

$$d(y, z) \leq d(y, z_0) + d(z_0, z) \leq \frac{u_n}{C} - r - 2CR + r = \frac{u_n}{C} - 2CR.$$

By Proposition 3.11-(vi), it follows that there is some $z' \in \overline{B}(y, u_n/C) \cap \text{im } \phi_n$ such that $d(z, z') \leq 2CR$. Then $d(z'_0, z') \leq 4CR + r$ by the triangle inequality. We have

$$d(z', \phi_n(S_n)) \geq d(z, \phi_n(S_n)) - d(z', z) > 2CR - 2CR = 0;$$

i.e., $z' \notin \phi_n(S_n)$. Let $\bar{z}'_0 = \phi_n^{-1}(z'_0) \in S_n$ and $\bar{z}' = \phi_n^{-1}(z') \in \mathcal{G}(x) \setminus S_n$. By Proposition 3.11-(v),

$$d(\bar{z}'_0, \bar{z}') \leq C d(z'_0, z') \leq C(4CR + r).$$

Thus

$$\bar{z}' \in \text{Pen}(S_n, C(4CR + r)) \setminus S_n \subset \partial_{C(4CR+r)} S_n.$$

So $z' \in \phi_n(\partial_{C(4CR+r)} S_n)$, and therefore $z \in \text{Pen}(\phi_n(\partial_{C(4CR+r)} S_n), 2CR)$.

Now take any $z \in \partial_r S_{y,n} \cap S_{y,n}$. Then there are points $z_0 \in \phi_n(S_n)$ and $z_1 \in \mathcal{G}(y) \setminus S_{y,n}$ such that $d(z, z_0) \leq 2CR$ and $d(z, z_1) \leq r$, obtaining $d(z_0, z_1) \leq 2CR + r$ by the triangle inequality. We have $d(z_1, \phi_n(S_n)) > 2CR$ because $z_1 \notin S_{y,n}$. By (4.9),

$$d(y, z_1) \leq d(y, z_0) + d(z_0, z_1) \leq \frac{u_n}{C} - r - 2CR + 2CR + r = \frac{u_n}{C}.$$

By Proposition 3.11-(vi), it follows that there is some $z'_1 \in \overline{B}(y, u_n/C) \cap \text{im } \phi_n$ such that $d(z_1, z'_1) \leq 2CR$. Then $d(z_0, z'_1) \leq 4CR + r$ by the triangle inequality. We have

$$d(z'_1, \phi_n(S_n)) \geq d(z_1, \phi_n(S_n)) - d(z'_1, z_1) > 2CR - 2CR = 0;$$

i.e., $z'_1 \notin \phi_n(S_n)$. Let $\bar{z}_0 = \phi_n^{-1}(z'_0) \in S_n$ and $\bar{z}'_1 = \phi_n^{-1}(z'_1) \in \mathcal{G}(x) \setminus S_n$. By Proposition 3.11-(v),

$$d(\bar{z}_0, \bar{z}'_1) \leq C d(z_0, z'_1) \leq C(4CR + r).$$

Thus

$$\bar{z}_0 \in \text{Pen}(\mathcal{G}(x) \setminus S_n, C(4CR + r)) \cap S_n \subset \partial_{C(4CR+r)} S_n.$$

So $z_0 \in \phi_n(\partial_{C(4CR+r)} S_n)$, and therefore $z \in \text{Pen}(\phi_n(\partial_{C(4CR+r)} S_n), 2CR)$. This completes the proof of Claim 12.

By Claim 12 and Proposition 3.11-(v),

$$(4.11) \quad |\partial_r S_{y,n}| \leq |\text{Pen}(\phi_n(\partial_{C(r+4CR)} S_n), 2CR)| \\ \leq \Lambda_{|E|, 2RC} |\phi_n(\partial_{C(r+4CR)} S_n)| = \Lambda_{|E|, 2RC} |\partial_{C(r+4CR)} S_n|;$$

in particular, by (2.16),

$$(4.12) \quad |\partial S_{y,n}| \leq \Lambda_{|E|, 2RC} |\partial_{C(1+4CR)} S_n| \leq \Lambda_{|E|, 2RC} \Lambda_{|E|, C(1+4CR)-1} |\partial S_n|.$$

Assume that \mathcal{G} is transitive. For every dense \mathcal{G} -orbit \mathcal{O} and all n , there is some $y_n \in \mathcal{O} \cap V(x, u_n)$. By (4.10), (4.12) and (2.16), the sets $S_{y_n, n}$ form a Følner sequence in \mathcal{O} , and therefore \mathcal{O} is amenable.

Assume that every dense \mathcal{G} -orbit is unbounded, for otherwise \mathcal{G} would have only one orbit. Then, for every dense \mathcal{G} -orbit \mathcal{O} , write

$$\mathcal{O} \cap V(x, u_n) = \{y(\mathcal{O}, m, n) \mid m \in \mathbb{N}\},$$

and let $S_{\mathcal{O}, m, n} = S_{y(\mathcal{O}, m, n), n}$.

Suppose that $\mathcal{O} \subset U_0$. Given m, n and $t \in \mathbb{N}$, and any \mathcal{G} -orbit \mathcal{O}' in $U_{0,d}$, let $v_n = u_n + C(2CR + t)$ and

$$\mathcal{N}_{\mathcal{O}, \mathcal{O}', m, n, t} = \{m' \in \mathbb{N} \mid y(\mathcal{O}', m', n) \in V(y(\mathcal{O}, m, n), v_n) \cap V(x, u_n)\},$$

which is a nonempty set. Take a Følner sequence X_n of \mathcal{O} such that $S_{\mathcal{O}, m, n} \subset X_n \subset \text{Pen}(S_{\mathcal{O}, m, n}, t)$. Thus, by (4.9),

$$X_n \subset \overline{B}(y(\mathcal{O}, m, n), u_n/C - r - 2CR + t) \\ = \overline{B}(y(\mathcal{O}, m, n), v_n/C - r - 4CR).$$

For the sake of simplicity, given n and any $m' \in \mathcal{N}_{\mathcal{O}', m, n}$, write $y = y(\mathcal{O}, m, n)$ and $y' = y(\mathcal{O}', m', n)$. Let

$$Y_{\mathcal{O}, \mathcal{O}', m', n} = \text{Pen}(\phi_{y, y', v_n}(X_n), 2CR) .$$

By Remark 3.12-(ii),

$$\begin{aligned} S_{\mathcal{O}', m', n} &= \text{Pen}(\phi_{x, y', u_n}(S_n), 2CR) = \text{Pen}(\phi_{y, y', v_n} \circ \phi_{x, y, u_n}(S_n), 2CR) \\ &\subset \text{Pen}(\phi_{y, y', v_n}(S_{\mathcal{O}, m, n}), 2CR) \subset \text{Pen}(\phi_{y, y', v_n}(X_n), 2CR) = Y_{\mathcal{O}, \mathcal{O}', m', n} , \end{aligned}$$

and, by (2.10), Proposition 3.11-(iv) and Remark 3.12,

$$\begin{aligned} Y_{\mathcal{O}, \mathcal{O}', m', n} &= \text{Pen}(\phi_{y, y', v_n}(X_n), 2CR) \subset \text{Pen}(\phi_{y, y', v_n}(\text{Pen}(S_{\mathcal{O}, m, n}, t)), 2CR) \\ &= \text{Pen}(\phi_{y, y', v_n}(\text{Pen}(\phi_{x, y, u_n}(S_n), 2CR + t)), 2CR) \\ &\subset \text{Pen}(\phi_{y, y', v_n} \circ \phi_{x, y, u_n}(S_n), 4CR + t) \\ &= \text{Pen}(\phi_{x, y', u_n}(S_n), 4CR + t) = \text{Pen}(S_{\mathcal{O}', m', n}, 2CR + t) . \end{aligned}$$

Furthermore, applying (4.10), (4.11) and (4.12) to X_n and $Y_{\mathcal{O}, \mathcal{O}', m', n}$, we get

$$\begin{aligned} |\partial_r Y_{\mathcal{O}, \mathcal{O}', m', n}| / |Y_{\mathcal{O}, \mathcal{O}', m', n}| &\leq \Lambda_{|E|, 2RC} |\partial_{C(r+4CR)} X_n| / |X_n| , \\ |\partial Y_{\mathcal{O}, \mathcal{O}', m', n}| / |Y_{\mathcal{O}, \mathcal{O}', m', n}| &\leq \Lambda_{|E|, 2RC} \Lambda_{|E|, C(1+4CR)-1} |\partial X_n| / |X_n| . \end{aligned}$$

This shows that the dense \mathcal{G} -orbits in U_0 are equi-amenable.

Next assume that \mathcal{G} is minimal. Then, by Proposition 3.6, for every \mathcal{O} and n , there is some $L_n \in \mathbb{N}$ so that $\mathcal{O} \cap V(x, u_n)$ is an L_n -net of \mathcal{O} for any \mathcal{G} -orbit \mathcal{O} . Then $\bigcup_m S_{\mathcal{O}, m, n}$ is an $(L_n + u_n/C - r - 2CR)$ -net in \mathcal{O} by (4.9). Similarly, for every \mathcal{O} , m , n and $t \in \mathbb{N}$, there is some $L_{\mathcal{O}, m, n, t} \in \mathbb{N}$ so that $\mathcal{O}' \cap V(y(\mathcal{O}, m, n), v_n) \cap V(x, u_n)$ is an $L_{\mathcal{O}, m, n, t}$ -net of \mathcal{O}' for any \mathcal{G} -orbit \mathcal{O}' . Then $\bigcup_{m' \in \mathcal{N}_{\mathcal{O}, \mathcal{O}', m, n, t}} S_{\mathcal{O}', m', n, t}$ is an $(L_{\mathcal{O}, m, n, t} + u_n/C - r - 2CR)$ -net in \mathcal{O}' by (4.9). This shows that, when \mathcal{G} is minimal, all \mathcal{G} -orbits in U_0 are jointly amenably symmetric. \square

4.4. Asymptotic dimension of the orbits

For $r \in \mathbb{Z}^+ \cup \{\infty\}$ and $R, D \in \mathbb{Z}^+$ and $n \in \mathbb{N}$, let $Y(r, R, D, n)$ be the set of elements $x \in U_0$ for which there exists families $\mathcal{V}_0, \dots, \mathcal{V}_n$ of subsets of the ball¹ $B(x, r)$ such that:

- (a) $\text{diam } V \leq D$ for all $V \in \mathcal{V}_i$;
- (b) $d(V, V') \geq R$ if $V \neq V'$ in \mathcal{V}_i ; and
- (c) $\bigcup_{i=0}^n \mathcal{V}_i$ covers $B(x, r)$.

We have

$$r \geq r', \ R \geq R', \ D \leq D' \implies Y(r, R, D, n) \subset Y(r', R', D', n) ;$$

in particular,

$$Y(R, D, n) := Y(\infty, R, D, n) \subset Y(r, R, D, n)$$

for all $r \in \mathbb{Z}^+$. To see the above inclusion, for a family \mathcal{V} of subsets of $B(x, r')$, consider the family $\mathcal{V}|_{B(x, r)}$ of intersections of the elements of \mathcal{V} with $B(x, r)$. Note that each set $Y(R, D, n)$ is saturated. Moreover, by Proposition 2.160,

$$(4.13) \quad \bigcap_{R, D} Y(R, D, n) = \{x \in U_0 \mid \text{asdim } \mathcal{G}(x) \leq n\} .$$

LEMMA 4.18. $Y(R, D, n) = \bigcap_r Y(r, R, D, n)$.

¹Recall that $B(x, \infty) = \mathcal{G}(x)$.

PROOF. Let $x \in \bigcap_r Y(r, R, D, n)$. Construct a graph with vertices the elements $(r, \mathcal{V}_0, \dots, \mathcal{V}_n)$, where $r \in \mathbb{Z}^+$ and $\mathcal{V}_0, \dots, \mathcal{V}_n$ are families of subsets of $B(x, r)$ satisfying (a)–(c) with these x and r , and having an edge from a vertex $(r, \mathcal{V}_0, \dots, \mathcal{V}_n)$ to another vertex $(r+1, \mathcal{W}_0, \dots, \mathcal{W}_n)$ if and only if $\mathcal{W}_i|_{B(x, r)} = \mathcal{V}_i$ for all $i \in \{0, \dots, n\}$. This graph is locally finite because $B(x, r)$ is finite for all r . On the other hand, the fact that $x \in \bigcap_r Y(r, R, D, n)$ implies that this graph has arbitrarily large rays. Therefore there is a sequence $r_k \uparrow \infty$, and, for each k , there are families $\mathcal{V}_{k,0}, \dots, \mathcal{V}_{k,n}$ of subsets of $B(x, r_k)$ satisfying (a)–(c) with x and r_k , and such that, whenever $k < l$, $\mathcal{V}_{k,i} = \mathcal{V}_{l,i}|_{B(x, r_k)}$ for all $i \in \{0, \dots, n\}$. Let \mathcal{V}_i be the family of unions $\bigcup_k \mathcal{V}_{k,i}$ for increasing sequences of sets, $V_0 \subset V_1 \subset \dots$, with $V_k \in \mathcal{V}_{k,i}$ for all k . It is easy to verify that the families $\mathcal{V}_0, \dots, \mathcal{V}_n$ satisfy (a)–(c) with x and $r = \infty$ (on $\mathcal{G}(x)$). Hence $x \in Y(R, D, n)$. \square

LEMMA 4.19. *For all $R, D \in \mathbb{Z}^+$ and $n \in \mathbb{N}$, there is some integer $D' \geq D$ so that $\text{Cl}_0(Y(R, D, n)) \subset Y(R, D', n)$.*

PROOF. With the notation of Proposition 3.11, for any $x \in \text{Cl}_0(Y(R, D, n))$, there are points $x_k \in Y(R, D, n) \cap V(x, k)$ for all $k \in \mathbb{Z}^+$ such that $x = \lim_k x_k$. According to Proposition 3.11, for some $C \in \mathbb{Z}^+$, independent of x , the maps $\phi_k := \phi_{x, x_k, k} : B(x, k) \rightarrow B(x_k, k)$ are non-expanding and equi-bi-Lipschitz with bi-Lipschitz distortion C for all k . For each k , take families $\mathcal{V}_{k,0}, \dots, \mathcal{V}_{k,n}$ of subsets of $B(x_k, k)$ satisfying (a)–(c) with x_k and $r = k$. For $k \geq k_0$, let

$$\mathcal{W}_{k,i} = \{ \phi_k^{-1}(V) \mid V \in \mathcal{V}_{k,i} \}$$

for each $i \in \{0, \dots, n\}$. Obviously, $\bigcup_{i=0}^n \mathcal{W}_{k,i}$ covers $B(x, k)$. Let $W = \phi_k^{-1}(V) \in \mathcal{W}_{k,i}$ with $V \in \mathcal{V}_{k,i}$. For $w, w' \in W$,

$$d(w, w') \leq C d(\phi_k(w), \phi_k(w')) \leq CD,$$

showing that $\text{diam } W \leq CD$. Take a different set $W' = \phi_k^{-1}(V') \in \mathcal{W}_{k,i}$ for $V' \neq V$ in $\mathcal{V}_{k,i}$. For $z \in W$ and $z' \in W'$,

$$d(z, z') \geq d(\phi_k(z), \phi_k(z')) \geq R,$$

obtaining that $d(W, W') \geq R$. So $x \in Y(k, R, CD, n)$ for all k , and therefore $x \in Y(R, CD, n)$ by Lemma 4.18. \square

COROLLARY 4.20. *For all $R \in \mathbb{Z}^+$ and $n \in \mathbb{N}$, $\bigcup_D Y(D, R, n)$ is an F_σ subset of U_0 .*

THEOREM 4.21. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. If \mathcal{G} is transitive, then residually many \mathcal{G} -orbits have the same asymptotic dimension.*

PROOF. The key step of the proof is the following assertion.

CLAIM 13. For each $n \in \mathbb{N}$, the set $\bigcap_R \bigcup_D Y(D, R, n)$ is either residual or meager in U .

If $\text{Int}_0(\bigcup_D Y(D, R, n)) \neq \emptyset$ for all R , then $\bigcup_D Y(D, R, n)$ is residual in U_0 , and therefore it is also residual in U , because this set is saturated and \mathcal{G} is transitive. Hence $\bigcap_R \bigcup_D Y(D, R, n)$ is also residual in U .

If $\text{Int}_0(\bigcup_D Y(D, R_0, n)) = \emptyset$ for some $R_0 \in \mathbb{Z}^+$, then $\bigcup_D Y(D, R_0, n)$ is meager in U_0 by Corollary 4.20, and therefore it is also meager in U , completing the proof of Claim 13.

Assume that $\bigcap_R \bigcup_D Y(D, R, n)$ is residual in U for some $n \in \mathbb{N}$, and let n_0 be the least n satisfying this property. By (4.13), n_0 is the asymptotic dimension of any \mathcal{G} -orbit in the \mathcal{G} -saturated set

$$(4.14) \quad \bigcap_R \bigcup_D Y(D, R, n_0) \setminus \bigcup_{n=0}^{n_0-1} \bigcap_R \bigcup_D Y(D, R, n),$$

which is residual by Claim 13.

Finally, suppose that there is no $n \in \mathbb{N}$ so that $\bigcap_R \bigcup_D Y(D, R, n)$ is residual in U . Hence $\bigcap_R \bigcup_D Y(D, R, n)$ is meager in U for all n by Claim 13, obtaining that

$$(4.15) \quad U_0 \setminus \bigcup_{n=0}^{\infty} \bigcap_R \bigcup_D Y(D, R, n)$$

is residual in U . Moreover every \mathcal{G} -orbit in this \mathcal{G} -saturated set is of infinite asymptotic dimension by (4.13). \square

4.5. Highson corona of the orbits

4.5.1. Limit sets. Let \mathcal{O} be an infinite \mathcal{G} -orbit, and $\overline{\mathcal{O}}$ a compactification of \mathcal{O} with corona $\partial\mathcal{O}$.

DEFINITION 4.22. The *limit set* of any subset $\Sigma \subset \partial\mathcal{O}$, denoted by $\lim_{\Sigma} \mathcal{O}$, is the subset $\bigcap_V \text{Cl}_U(V \cap \mathcal{O})$ of U , where V runs in the collection of neighborhoods of Σ in $\overline{\mathcal{O}}$. If $\Sigma = \{e\}$, then the notation $\lim_e \mathcal{O}$ is used for $\lim_{\Sigma} \mathcal{O}$.

Take another compactification $\overline{\mathcal{O}}' \leq \overline{\mathcal{O}}$ of \mathcal{O} with corona $\partial'\mathcal{O}$. Thus there is a continuous extension $\pi : \overline{\mathcal{O}} \rightarrow \overline{\mathcal{O}}'$ of $\text{id}_{\mathcal{O}}$. The restriction $\pi : \partial\mathcal{O} \rightarrow \partial'\mathcal{O}$ clearly satisfies $\lim_e \mathcal{O} \subset \lim_{\pi(e)} \mathcal{O}$ for all $e \in \partial\mathcal{O}$. Thus, roughly speaking, smaller compactifications of the orbits induce larger limit sets.

Each limit set is a closed subset of U , which may or may not be \mathcal{G} -saturated. The following examples will serve as illustration of this fact.

- EXAMPLES 4.23. (i) The corona of the one-point compactification \mathcal{O}^* is a singleton, and the corresponding limit set is the standard limit set of the orbit \mathcal{O} , which of course \mathcal{G} -saturated.
- (ii) Consider the compactification of \mathcal{O} whose corona is its coarse end space. The limit set of \mathcal{O} at any of its coarse ends is \mathcal{G} -saturated.
- (iii) As a particular case of (ii), if $Z = U$ is compact, $\mathcal{H} = \mathcal{G}$ is the pseudogroup generated by a homeomorphism h of Z , and $E = \{h^{\pm 1}\}$, then \mathcal{O} is isometric to \mathbb{Z} , whose coarse end space consists of two points. The corresponding limit sets are the usual α - and ω -limits of \mathcal{O} , which are \mathcal{G} -saturated.
- (iv) Consider the set

$$\overline{\mathcal{O}} = \mathcal{O} \sqcup \text{Cl}_Z(\mathcal{O}) = (\mathcal{O} \times \{0\}) \cup (\text{Cl}_Z(\mathcal{O}) \times \{1\})$$

with the topology determined as follows: each point of $\mathcal{O} \times \{0\}$ is isolated in $\overline{\mathcal{O}}$, and, a basic neighborhood of a point in $(z, 1) \in \text{Cl}_Z(\mathcal{O}) \times \{1\}$ in $\overline{\mathcal{O}}$ is of the form $(V \cap \mathcal{O}) \sqcup V$, where V is any neighborhood of z in $\text{Cl}_Z(U)$. Observe that $\overline{\mathcal{O}}$ is a compact Hausdorff space, and $\mathcal{O} \equiv \mathcal{O} \times \{0\}$ is open and dense in $\overline{\mathcal{O}}$; thus $\overline{\mathcal{O}}$ is a compactification of \mathcal{O} . In terms of algebras of functions, $\overline{\mathcal{O}}$ corresponds to the algebra of \mathbb{C} -valued functions on \mathcal{O} that admit a continuous extension to $\text{Cl}_Z(\mathcal{O})$. The corona of this compactification is $\partial\mathcal{O} = \text{Cl}_Z(\mathcal{O}) \times \{1\} \equiv \text{Cl}_Z(\mathcal{O})$. Moreover, for each $z \in \partial\mathcal{O}$, it is easy to see that $\lim_z \mathcal{O} = \{z\}$ if $z \in U$, and $\lim_z \mathcal{O} = \emptyset$ if $z \notin U$. Thus $\lim_z \mathcal{O}$ may not be \mathcal{G} -saturated.

- (v) For the Stone-Ćech compactification \mathcal{O}^β , the limit set of \mathcal{O} at any point in its corona $\beta\mathcal{O}$ is either a singleton or empty by (iv) since \mathcal{O}^β is the maximum of the compactifications.
- (vi) For any compactification $\overline{\mathcal{O}} \leq \mathcal{O}^\nu$ with corona $\partial\mathcal{O}$, it will be shown that the limit sets of \mathcal{O} at points in $\partial\mathcal{O}$ are \mathcal{G} -saturated (Theorem 4.25).
- (vii) As a particular case of (vi), if (\mathcal{O}, d_E) is hyperbolic (in the sense of Gromov), we can consider its compactification whose corona is the ideal boundary. Then the limit sets of \mathcal{O} at points in its ideal boundary are \mathcal{G} -saturated.

LEMMA 4.24. *Let $x \in \lim_\Sigma \mathcal{O}$ for some $\Sigma \subset \partial\mathcal{O}$. If $\overline{\mathcal{O}} \leq \mathcal{O}^\nu$, V is a neighborhood of Σ in $\overline{\mathcal{O}}$, and $S_1 \subset S_2 \subset \dots$ is an increasing sequence of bounded subsets of $V \cap \mathcal{O}$, then there is a sequence $x_i \rightarrow x$ in U such that*

$$\overline{B}(x_i, i) \subset V \cap \mathcal{O}, \quad d(x_i, \{x_1, \dots, x_{i-1}\} \cup S_i) > 3i.$$

PROOF. Since $\mathcal{O} \leq \mathcal{O}^\nu$, there is a continuous extension $\pi : \mathcal{O}^\nu \rightarrow \overline{\mathcal{O}}$ of $\text{id}_\mathcal{O}$. The sets $\tilde{V} := \pi^{-1}(V)$, for neighborhoods V of Σ in $\overline{\mathcal{O}}$, form a base of neighborhoods of $\tilde{\Sigma} := \pi^{-1}(\Sigma)$ in \mathcal{O}^ν , and we have $\tilde{V} \cap \mathcal{O} = V \cap \mathcal{O}$, obtaining also $\lim_{\tilde{\Sigma}} \mathcal{O} = \lim_\Sigma \mathcal{O}$. So it is enough to consider only the case where $\overline{\mathcal{O}} = \mathcal{O}^\nu$.

Assuming $\overline{\mathcal{O}} = \mathcal{O}^\nu$, let $W = V \cap \mathcal{O}$, and, for $i \in \mathbb{Z}^+$, let

$$W_i = \{y \in W \mid d(y, \mathcal{O} \setminus W) > i\}, \quad V_i = \text{Int}_{\mathcal{O}^\nu}(\text{Cl}_{M^\nu}(W_i)).$$

Observe that $W_i = V_i \cap \mathcal{O}$ because \mathcal{O} is a discrete space. By Proposition 2.154, V_i is a neighborhood of Σ in \mathcal{O}^ν , and therefore $x \in \text{Cl}_U(W_i)$. Then, given a countable base $\{P_i\}$ of open neighborhoods of x in U , we have $P_i \cap W_k \neq \emptyset$ for all i and k .

The elements x_i are defined by induction on i . Let

$$l_1 = \max\{d(y, \mathcal{O} \setminus W) \mid y \in S_1\}.$$

We can choose any element $x_1 \in P_1 \cap W_{3+l_1}$. Then, for all $y \in S_1$,

$$d(x_1, y) \geq d(x_1, \mathcal{O} \setminus W) - d(y, \mathcal{O} \setminus W) > 3 + l_1 - d(y, \mathcal{O} \setminus W) \geq 3.$$

Now let $i > 1$ and assume that x_j is defined for all $j < i$ satisfying $x_j \in P_j \cap W_j$ and $d(x_j, \{x_1, \dots, x_{j-1}\} \cup S_j) > 3j$. Let

$$l_i = \max\{d(y, \mathcal{O} \setminus W) \mid y \in \{x_1, \dots, x_{i-1}\} \cup S_i\},$$

and choose any point $x_i \in P_i \cap W_{3+l_i}$. For all $y \in \{x_1, \dots, x_{i-1}\} \cup S_i$,

$$d(x_i, y) \geq d(x_i, \mathcal{O} \setminus W) - d(y, \mathcal{O} \setminus W) > 3i + l_i - d(y, \mathcal{O} \setminus W) \geq 3i.$$

Moreover $\overline{B}(x_i, i) \subset W$ because $x_i \in W_{3+l_i} \subset W_i$. □

THEOREM 4.25. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1, and let $\overline{\mathcal{O}}$ be a compactification of a \mathcal{G} -orbit \mathcal{O} , with corona $\partial\mathcal{O}$. If $\overline{\mathcal{O}} \leq \mathcal{O}^\nu$, then $\lim_{\mathbf{e}} \mathcal{O}$ is \mathcal{G} -saturated and nonempty for all $\mathbf{e} \in \partial\mathcal{O}$.*

PROOF. Let $x \in \lim_{\mathbf{e}} \mathcal{O}$ for some \mathcal{G} -orbit \mathcal{O} and $\mathbf{e} \in \partial\mathcal{O}$. Take a sequence $x_i \rightarrow x$ satisfying the conditions of Lemma 4.24 with any neighborhood V of \mathbf{e} in $\overline{\mathcal{O}}$ and $S_i = \emptyset$ for all i . Then, by Proposition 3.16, for any $r \in \mathbb{Z}^+$,

$$\overline{B}(x, r) \subset \bigcap_{i>r} \text{Cl}_U \left(\bigcup_{j \geq i} \overline{B}(x_j, r) \right) \subset \text{Cl}_U(V \cap \mathcal{O}).$$

Since V and r are arbitrary, it follows that $\mathcal{G}(x) \subset \lim_{\mathbf{e}} \mathcal{O}$. Hence $\lim_{\mathbf{e}} \mathcal{O}$ is saturated.

Let U_0 be a relatively compact open subset of U that meets all \mathcal{G} -orbits. Since, for any open neighborhood V of \mathbf{e} in $\overline{\mathcal{O}}$, the set $V \cap \mathcal{O}$ contains balls of arbitrarily large radius, we have $U_0 \cap V \neq \emptyset$ by Proposition 3.6, and therefore $\text{Cl}_U(V \cap \mathcal{O}) \cap \text{Cl}_U(U_0)$ is a nonempty compact set. It follows that

$$\lim_{\mathbf{e}} \mathcal{O} \cap \text{Cl}_U(U_0) = \bigcap_V \text{Cl}_U(V \cap \mathcal{O}) \cap \text{Cl}_U(U_0) \neq \emptyset,$$

showing that $\lim_{\mathbf{e}} \mathcal{O} \neq \emptyset$. \square

DEFINITION 4.26. A \mathcal{G} -orbit \mathcal{O} is said to be *Higson recurrent* if $\lim_{\mathbf{e}} \mathcal{O} = U$ for all $\mathbf{e} \in \nu\mathcal{O}$.

REMARK 4.27. Every Higson recurrent \mathcal{G} -orbit is obviously dense in U .

THEOREM 4.28. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. A \mathcal{G} -orbit is Higson recurrent if and only if \mathcal{G} is minimal.*

PROOF. Let \mathcal{O} be a Higson recurrent \mathcal{G} -orbit, and let Y be a \mathcal{G} -minimal set (Proposition 3.24). Since \mathcal{O} is dense (Remark 4.27), there is a convergent sequence in U , $x_i \rightarrow x$, with $x_i \in \mathcal{O}$ and $x \in Y$. Let $P_1 \supset P_2 \supset \dots$ be a nested sequence of open neighborhoods of Y in U such that $\bigcap_k \text{Cl}_U(P_k) = Y$. By Proposition 3.16, for each $k \in \mathbb{Z}^+$, there is some index i_k such that $\overline{B}(x_{i_k}, k) \subset P_k$. Hence

$$\bigcap_l \text{Cl}_U \left(\bigcup_{k \geq l} \overline{B}(x_{i_k}, k) \right) \subset \bigcap_l \text{Cl}_U(P_l) = Y.$$

By Proposition 2.153, $W = \bigcup_k B(x_{i_k}, k) \subset \mathcal{O}$ satisfies that $V = \text{Int}_{\mathcal{O}^\nu}(\text{Cl}_{\mathcal{O}^\nu}(W))$ is an open neighborhood of some $\mathbf{e} \in \nu\mathcal{O}$ in \mathcal{O}^ν . By Proposition 2.154, the set $V_l = V \setminus \bigcup_{k=1}^l \overline{B}(x_{i_k}, k)$ is another open neighborhood of \mathbf{e} in \mathcal{O}^ν . Since \mathcal{O} is Higson recurrent, we get

$$U = \lim_{\mathbf{e}} \mathcal{O} \subset \bigcap_l \text{Cl}_U(V_l \cap \mathcal{O}) = \bigcap_l \text{Cl}_U \left(\bigcup_{k \geq l} \overline{B}(x_{i_k}, k) \right) \subset Y.$$

Thus U is the only \mathcal{G} -minimal set; i.e., \mathcal{G} is minimal.

Now, assume that \mathcal{G} is minimal. Then $\lim_{\mathbf{e}} \mathcal{O} = U$ for all \mathcal{G} -orbit \mathcal{O} and $\mathbf{e} \in \mathcal{O}^\nu$ because $\lim_{\mathbf{e}} \mathcal{O}$ is a \mathcal{G} -saturated non-empty closed subset of U (Theorem 4.25). \square

For each \mathcal{G} -minimal set Y and any \mathcal{G} -orbit \mathcal{O} , let $\nu_Y \mathcal{O} = \{\mathbf{e} \in \nu\mathcal{O} \mid \lim_{\mathbf{e}} \mathcal{O} = Y\}$.

THEOREM 4.29. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. For any \mathcal{G} -orbit \mathcal{O} , the set $\bigcup_Y \text{Int}_{\nu\mathcal{O}}(\nu_Y \mathcal{O})$, where Y runs in the family of \mathcal{G} -minimal sets, is dense in $\nu\mathcal{O}$.*

PROOF. Let $\mathbf{e} \in \nu\mathcal{O}$ for some \mathcal{G} -orbit \mathcal{O} , and let V be a neighborhood of \mathbf{e} in \mathcal{O}^ν . Take another open neighborhood V' of \mathbf{e} in \mathcal{O}^ν such that $\text{Cl}_{\mathcal{O}^\nu}(V') \subset V$. By Proposition 3.24 and Theorem 4.25, $\lim_{\mathbf{e}} \mathcal{O}$ contains a \mathcal{G} -minimal set Y . Like in the proof of Theorem 4.28, we can find a subset $W \subset V' \cap \mathcal{O}$ so that, for $V'' = \text{Int}_{\mathcal{O}^\nu}(\text{Cl}_{\mathcal{O}^\nu}(W))$, any $\mathbf{e}' \in V'' \cap \nu\mathcal{O}$ satisfies $\lim_{\mathbf{e}'} \mathcal{O} \subset Y$, and therefore $\lim_{\mathbf{e}'} \mathcal{O} = Y$ because Y is a minimal set. Thus $V'' \cap \nu\mathcal{O} \subset \nu_Y \mathcal{O}$ and $V'' \subset \text{Cl}_{\mathcal{O}^\nu}(V') \subset V$. \square

Suppose that $(Z', \mathcal{H}', U', \mathcal{G}', E')$ also satisfies Hypothesis 1, and that there is an equivalence $\mathcal{H} \rightarrow \mathcal{H}'$, which induces an equivalence $\mathcal{G} \rightarrow \mathcal{G}'$, an homeomorphism $U/\mathcal{G} \rightarrow U'/\mathcal{G}'$, and a bijection between the families of \mathcal{G} - and \mathcal{G}' -saturated sets. For each \mathcal{G} -orbit \mathcal{O} , let \mathcal{O}' denote the corresponding \mathcal{G}' -orbit. By Theorem 3.8, there are equi-coarse quasi-isometries of the metric spaces (\mathcal{O}, d_E) to the corresponding

metric spaces $(\mathcal{O}', d_{E'})$. By Propositions 2.26, 2.39 and 2.150, these equi-coarse quasi-isometries induce maps between the corresponding Higson compactifications, $\mathcal{O}^\nu \rightarrow \mathcal{O}'^\nu$, that are continuous at the points of the Higson coronas, and restrict to homeomorphisms between the corresponding Higson coronas, $\nu\mathcal{O} \rightarrow \nu\mathcal{O}'$. In fact, the maps $\mathcal{O}^\nu \rightarrow \mathcal{O}'^\nu$ are continuous at all points because the orbits are discrete metric spaces.

PROPOSITION 4.30. *For corresponding orbits, \mathcal{O} of \mathcal{G} and \mathcal{O}' of \mathcal{G}' , and corresponding points $\mathbf{e} \in \nu\mathcal{O}$ and $\mathbf{e}' \in \nu\mathcal{O}'$, the \mathcal{G} -saturated set $\lim_{\mathbf{e}} \mathcal{O}$ corresponds to the \mathcal{G}' -saturated set $\lim_{\mathbf{e}'} \mathcal{O}'$.*

PROOF. By Remark 3.3, it is enough to consider the case where $Z' = Z$, $\mathcal{H}' = \mathcal{H}$, $\text{Cl}_Z(U) \subset U'$, and $E' = \overline{E} = \{\bar{g} \mid g \in E\}$, where \bar{g} is an extension of each $g \in E$ with $\text{Cl}_Z(\text{dom } g) \subset \text{dom } \bar{g}$ (like in Section 3.2). Then $\mathcal{O} = \mathcal{O}' \cap U$ and $\mathcal{O}' = \mathcal{G}'(\mathcal{O})$. Moreover the above coarse quasi-isometry of (\mathcal{O}, d_E) to $(\mathcal{O}', d_{\overline{E}})$ is the inclusion map $\mathcal{O} \hookrightarrow \mathcal{O}'$, which is a C -bi-Lipschitz map whose image is an R -net with respect to the metrics d_E and $d_{\overline{E}}$ (see Section 3.2). It induces the above continuous map $\mathcal{O}^\nu \rightarrow \mathcal{O}'^\nu$, which is an embedding in this case (Corollary 2.152). Thus we will consider \mathcal{O}^ν as a subspace of \mathcal{O}'^ν with $\nu\mathcal{O} = \nu\mathcal{O}'$; in particular, $\mathbf{e} = \mathbf{e}'$.

Let V' be an arbitrary open neighborhood of \mathbf{e} in \mathcal{O}'^ν , and therefore $V = V' \cap \mathcal{O}^\nu$ is an arbitrary open neighborhood of \mathbf{e} in \mathcal{O}^ν . We have $V \cap \mathcal{O} = V' \cap \mathcal{O}' \cap U$. So

$$\text{Cl}_U(V \cap \mathcal{O}) = \text{Cl}_U(V' \cap \mathcal{O}' \cap U) = \text{Cl}_{U'}(V' \cap \mathcal{O}') \cap U,$$

where the inclusion “ \supset ” of last equality holds because U is open in U' . It follows that

$$\lim_{\mathbf{e}} \mathcal{O} = \bigcap_V \text{Cl}_U(V \cap \mathcal{O}) = \bigcap_{V'} \text{Cl}_{U'}(V' \cap \mathcal{O}') \cap U = U \cap \lim_{\mathbf{e}} \mathcal{O}'. \quad \square$$

4.5.2. Semi weak homogeneity of the Higson corona.

DEFINITION 4.31. A topological space X is called *semi weakly homogeneous* if, for all nonempty open subsets $V, V' \subset X$, there are homeomorphic nonempty open subsets, $\Omega \subset V$ and $\Omega' \subset V'$.

THEOREM 4.32. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1. If \mathcal{G} is minimal, then the space $\bigsqcup_{\mathcal{O}} \nu\mathcal{O}$, with \mathcal{O} running in the set of all \mathcal{G} -orbits in U_0 , is semi weakly homogeneous.*

PROOF. Let \mathcal{O} and \mathcal{O}' be \mathcal{G} -orbits in U_0 , and let $\mathbf{e} \in \nu\mathcal{O}$ and $\mathbf{e}' \in \nu\mathcal{O}'$. Given open neighborhoods, V of \mathbf{e} in \mathcal{O}^ν and V' of \mathbf{e}' in \mathcal{O}'^ν , take other open neighborhoods, V_0 of \mathbf{e} in \mathcal{O}^ν and V'_0 of \mathbf{e}' in \mathcal{O}'^ν , such that $\text{Cl}_{\mathcal{O}^\nu}(V_0) \subset V$ and $\text{Cl}_{\mathcal{O}'^\nu}(V'_0) \subset V'$. By Corollary 2.155, there is a sequence x_k in \mathcal{O} such that $\overline{B}(x_k, k) \subset V_0$ and $d(x_l, x_k) > 3k$ if $l < k$. We have $\lim_{\mathbf{e}'} \mathcal{O}' = U$ by Theorem 4.28. Using Lemma 4.24 by induction on k , it follows that there are convergent sequences in U , $x'_{k,i} \rightarrow x_k$, such that $B(x'_{k,i}, i) \subset V'_0 \cap \mathcal{O}'$, and $d(x'_{l,i}, x'_{k,j}) > 3i$ if $l < k$ and $j \leq i$, or $l = k$ and $j < i$. With the notation of Proposition 3.11, for all $k \in \mathbb{Z}^+$, there is some increasing sequence of indices i_k such that $i_k \geq k$ and $x'_{k,i_k} \in V(x_k, k)$.

According to Proposition 3.11-(v),(vi), the restrictions

$$\phi_k := \phi_{x_k, x'_{k,i_k}, k} : B_k := \phi_k^{-1}(\overline{B}(x'_{k,i_k}, k/C)) \rightarrow B'_k := \overline{B}(x'_{k,i_k}, k/C),$$

for $k \geq CR$, form a family of equi-coarse equivalences. Moreover $B_k \subset \overline{B}(x_k, k)$ and $B'_k \subset \overline{B}(x'_{k,i_k}, k) \subset \overline{B}(x'_{k,i_k}, i_k)$, obtaining $d(B_k, B_l) > k$ and $d(B'_k, B'_l) > i_k \geq k$ if $l < k$.

k . Then, by Proposition 2.44, the combination of the maps ϕ_k is coarse equivalence $\phi : B := \bigcup_{k \geq k_0} B_k \rightarrow B' := \bigcup_{k \geq k_0} B'_k$. Thus $\nu\phi : \nu B \rightarrow \nu B'$ is a homeomorphism by Proposition 2.150-(iii). We have canonical identities $\nu B \equiv \text{Cl}_{\mathcal{O}^\nu}(B) \cap \nu\mathcal{O}$ and $\nu B' \equiv \text{Cl}_{\mathcal{O}^\nu}(B') \cap \nu\mathcal{O}'$ (Corollary 2.152). Let $V_1 = \text{Int}_{\mathcal{O}^\nu}(\text{Cl}_{\mathcal{O}^\nu}(B))$ and $V'_1 = \text{Int}_{\mathcal{O}^\nu}(\text{Cl}_{\mathcal{O}^\nu}(B'))$. The open subsets $\Omega_1 := V_1 \cap \nu\mathcal{O} \subset \nu\mathcal{O}$ and $\Omega'_1 := V'_1 \cap \nu\mathcal{O}' \subset \nu\mathcal{O}'$ are nonempty by Proposition 2.153 since $B_k \supset \overline{B}(x_k, k/C)$ because ϕ_k is non-expanding (Proposition 3.11-(iv)). By Corollary 2.156, the sets Ω_1 and Ω'_1 are also dense in νB and $\nu B'$, respectively. Hence $\Omega := \Omega_1 \cap (\nu\phi)^{-1}(\Omega'_1)$ and $\Omega' := \nu\phi(\Omega_1) \cap \Omega'_1$ are open dense subsets of Ω_1 and Ω'_1 , respectively, and therefore Ω and Ω' are nonempty and open in νM . Moreover $\nu\phi$ restricts to a homeomorphism $\Omega \rightarrow \Omega'$. Finally,

$$\text{Cl}_{\nu\mathcal{O}}(\Omega_1) = \text{Cl}_{\mathcal{O}^\nu}(V_1) \cap \nu\mathcal{O} \subset \text{Cl}_{\mathcal{O}^\nu}(B) \cap \nu\mathcal{O} \subset \text{Cl}_{\mathcal{O}^\nu}(V_0) \cap \nu\mathcal{O} \subset V \cap \nu\mathcal{O},$$

and, similarly, $\text{Cl}_{\nu\mathcal{O}'}(\Omega'_1) \subset V' \cap \nu\mathcal{O}'$. \square

4.6. Measure theoretic versions

Let μ be a Borel measure on Z . The *measure class* $[\mu]$ on Z is the set of Borel measures on Z with the same sets of zero measure as μ , and therefore also the same sets of full measure (the complements of the sets of zero measure). If a measurable $A \subset Z$ is of full measure, then $[\mu]$ is said to be *supported* in A . The *product measure class* $[\mu] \times [\mu]$ on $Z \times Z$ is the measure class represented by the product measure $\mu \times \mu$.

Let μ a measure on Z . The measure class $[\mu]$ is \mathcal{H} -invariant if $\mu(B) = 0 \Rightarrow \mu(h(B)) = 0$, for all $h \in \mathcal{H}$ and every measurable $B \subset \text{dom } h$. An \mathcal{H} -invariant measure class $[\mu]$ is said to be *ergodic* (or \mathcal{H} -*ergodic*) when it consists of ergodic measures; i.e., any \mathcal{H} -saturated measurable set is either of zero measure, or of full measure for every measure in $[\mu]$. In this case, $[\mu] \times [\mu]$ is also \mathcal{H} -invariant and \mathcal{H} -ergodic.

LEMMA 4.33. *Let X be a topological space, and $A \subset X$ a G_δ subset. Then every Borel subset of A is Borel in X .*

PROOF. It is enough to consider the case of an open subset $B \subset A$, that is, $B = A \cap V$ for some open $V \subset X$. Because A is G_δ in X , it can be expressed as $A = \bigcap_n U_n$, for countable many open sets $U_n \subset X$. Therefore $B = \bigcap_n (U_n \cap V)$ is a G_δ subset of X . \square

HYPOTHESIS 2. A sextuple $(Z, \mathcal{H}, U, \mathcal{G}, E, [\mu])$ is required to satisfy the following conditions:

- $(Z, \mathcal{H}, U, \mathcal{G}, E)$ satisfy Hypothesis 1, and
- $[\mu]$ is a \mathcal{G} -invariant measure class on U supported in U_0 .

THEOREM 4.34. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E, [\mu])$ satisfy Hypothesis 2. If $[\mu]$ is \mathcal{G} -ergodic, then:*

- (i) *either $[\mu]$ -almost all \mathcal{G} -orbits are coarsely quasi-isometric to $[\mu]$ -almost all \mathcal{G} -orbits;*
- (ii) *or else $[\mu]$ -almost all \mathcal{G} -orbits are coarsely quasi-isometric to $[\mu]$ -almost no \mathcal{G} -orbit.*

PROOF. Consider the notation of Section 4.1. In particular, Y is a $\mathcal{G} \times \mathcal{G}$ -saturated Borel subset of $U_0 \times U_0$ by Lemma 4.1, and therefore it is also a Borel subset of $U \times U$ by Lemma 4.33. By the $\mathcal{G} \times \mathcal{G}$ -ergodicity of the product measure class on $U \times U$, either Y is of full measure, or Y is of zero measure.

Suppose Y is of full measure. By Fubini's theorem, the set $Y_x = \{y \in U_0 \mid (x, y) \in Y\}$ is of full measure for almost all $x \in U$ and (i) obtains.

Now assume that Y is of measure zero. It follows from Fubini's theorem that Y_x is of zero measure for almost all $x \in U$, and (ii) obtains. \square

THEOREM 4.35. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E, [\mu])$ satisfy Hypothesis 2. If $[\mu]$ is \mathcal{G} -ergodic, then:*

- (i) *either $[\mu]$ -almost all \mathcal{G} -orbits have the same growth type;*
- (ii) *or else the growth type of $[\mu]$ -almost all \mathcal{G} -orbits are comparable with the growth type of $[\mu]$ -almost no \mathcal{G} -orbit.*

PROOF. With the notation of Section 4.2, Y is a $\mathcal{G} \times \mathcal{G}$ -saturated Borel subset of $U_0 \times U_0$ by Lemma 4.10, and therefore it is also a Borel subset of $U \times U$ by Lemma 4.33. Like in the proof of Theorem 4.34, it follows that, either Y is of full measure, or Y is of zero measure. Hence, either Y_τ is of full measure, or Y^τ is of zero measure. Then the result follows with the arguments of the proof of Theorem 4.34, using Y_τ to get (i), and Y^τ to get (ii). \square

THEOREM 4.36. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E, [\mu])$ satisfy Hypothesis 2. If $[\mu]$ is \mathcal{G} -ergodic, then the equalities and inequalities of Theorem 4.14 hold $[\mu]$ -almost everywhere with some $a_1, a_3 \in [1, \infty]$, $a_2, a_4 \in [0, \infty]$ and $p \geq 1$.*

PROOF. Consider the notation of Theorem 4.14. Its proof shows that the sets $Y(a)$, $\mathcal{G}(Y'_2(a))$, $Y_3(a)$ and $\mathcal{G}(Y_4(a))$ are \mathcal{G} -invariant and Borel in U_0 for all $a \in [0, \infty]$. Thus they are also Borel in U by Lemma 4.33. By ergodicity, each of these sets are either of zero measure or of full measure. Then the result follows easily from the definition of these sets, and using (4.2), (4.3), (4.5) and (4.6) like in the proof of Theorem 4.14. \square

COROLLARY 4.37. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E, [\mu])$ satisfy Hypothesis 2. If $[\mu]$ is \mathcal{G} -ergodic, then any of the sets of Corollary 4.15 is either of zero $[\mu]$ -measure or of full $[\mu]$ -measure.*

THEOREM 4.38. *Let $(Z, \mathcal{H}, U, \mathcal{G}, E, [\mu])$ satisfy Hypothesis 2. If $[\mu]$ is \mathcal{G} -ergodic, then $[\mu]$ -almost all \mathcal{G} -orbits have the same asymptotic dimension.*

PROOF. Consider the notation of Theorem 4.21. By Lemma 4.20, each \mathcal{G} -saturated set $\cap_R \cup_D Y(D, R, n)$ is Borel. So, by ergodicity, this set is, either of full $[\mu]$ -measure, or of zero $[\mu]$ -measure.

Suppose that $\cap_R \cup_D Y(D, R, n)$ is of full $[\mu]$ -measure for some n , and let n_0 be the least n satisfying this property. Like in the proof of Theorem 4.21, n_0 is the asymptotic dimension of any \mathcal{G} -orbit in the \mathcal{G} -saturated set (4.14), which is of full $[\mu]$ -measure.

If $\cap_R \cup_D Y(D, R, n)$ is of zero $[\mu]$ -measure for all n , then, like in the proof of Theorem 4.21, the \mathcal{G} -saturated set (4.15) is of full $[\mu]$ -measure, and consists of orbits with infinite asymptotic dimension. \square

CHAPTER 5

Generic coarse geometry of leaves

This chapter is devoted to recall preliminaries needed about foliated spaces, fixing the notation, so that the main theorems follow directly from their pseudogroup versions. Introductions to foliated spaces, with many examples, are given in [MS88], [CC00, Chapter 11], [CC03, Part 1] and [Ghy00].

5.1. Foliated spaces

Let Z be a space and let U be an open set in $\mathbb{R}^n \times Z$ ($n \in \mathbb{N}$), with coordinates (x, z) . For $m \in \mathbb{N}$, a map $f : U \rightarrow \mathbb{R}^p$ ($p \in \mathbb{N}$) is (*smooth* or *differentiable*) of *class* C^m if its partial derivatives up to order m with respect to x exist and are continuous on U . If f is of class C^m for all m , then it is called (*smooth* or *differentiable*) of *class* C^∞ .

Let Z' be another space, and let $h : U \rightarrow \mathbb{R}^p \times Z'$ be a map of the form $h(x, z) = (h_1(x, z), h_2(z))$, for maps $h_1 : U \rightarrow \mathbb{R}^p$ and $h_2 : \text{pr}_2(U) \rightarrow Z'$. It will be said that h is of *class* C^m if h_1 is of class C^m and h_2 is continuous.

For $m \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$, a *foliated structure*¹ \mathcal{F} of *class* C^m and *dimension* $\dim \mathcal{F} = n$ on a space X is defined by a collection $\mathcal{U} = \{U_i, \phi_i\}$, where $\{U_i\}$ is an open covering of X , and each ϕ_i is a homeomorphism $U_i \rightarrow B_i \times Z_i$, for a locally compact Polish space Z_i and an open ball B_i in \mathbb{R}^n , such that the coordinate changes $\phi_j \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ are locally C^m maps of the form

$$(5.1) \quad \phi_j \phi_i^{-1}(x, z) = (x_{ij}(x, z), h_{ij}(z)) .$$

Each (U_i, ϕ_i) is called a *foliated chart* or *flow box*, the sets $\phi_i^{-1}(B_i \times \{z\})$ ($z \in Z_i$) are called *plaques* (or *\mathcal{U} -plaques*), and the collection \mathcal{U} is called a *foliated atlas* (of *class* C^m). Two C^m foliated atlases on X define the same C^m *foliated structure* if their union is a C^m foliated atlas. If we consider foliated atlases so that the sets Z_i are open in some fixed space, then \mathcal{F} can be also described as a maximal foliated atlas of class C^m . The term *foliated space* (of *class* C^m) is used for $X \equiv (X, \mathcal{F})$. If no reference to the class C^m is indicated, then it is understood that X is a C^0 (or *topological*) foliated space. The restriction of \mathcal{F} to some open subset $U \subset X$ is the foliated structure $\mathcal{F}|_U$ on U defined by the charts of \mathcal{F} whose domains are contained in U .

A map between foliated spaces is called *foliated* if it maps leaves to leaves. A foliated map between C^m foliated spaces is said to be of *class* C^m if its local representations in terms of foliated charts are of class C^m .

¹The term *lamination* is also used, specially when X is a subspace of a manifold. The term *foliation* is used when the spaces Z_i are open subsets of some Euclidean space, and therefore X is a manifold. The condition to be of *class* C^m for a foliation \mathcal{F} also requires that the maps $\phi_j \phi_i^{-1}$ are of class C^m .

The foliated structure of a space X induces a locally Euclidean topology on X , the basic open sets being the plaques of all of its foliated charts, which is finer than the original topology. The connected components of X in this topology are called *leaves* (or \mathcal{F} -*leaves*). Each leaf becomes a connected manifold of dimension n and of class C^m with the differential structure canonically induced by \mathcal{F} . The leaf which contains each point $x \in X$ will usually be denoted by L_x . The leaves of \mathcal{F} form a partition of X that determines the (topological) foliated structure. The corresponding quotient space, called *leaf space*, is denoted by X/\mathcal{F} . It is said that \mathcal{F} is *transitive* (respectively, *minimal*) when some leaf is dense (respectively, all leaves are dense) in X .

Many concepts of manifold theory readily extend to foliated spaces. In particular, if \mathcal{F} is of class C^m with $m \geq 1$, there is a vector bundle $T\mathcal{F}$ over X whose fiber at each point $x \in X$ is the tangent space $T_x L_x$. Observe that $T\mathcal{F}$ is a foliated space of class C^{m-1} with leaves TL for leaves L of X , and any section of $T\mathcal{F}$ is foliated. The same applies to any bundle naturally associated to $T\mathcal{F}$. Then we can consider a C^{m-1} Riemannian structure² on $T\mathcal{F}$, which will be called a (*leafwise*) *Riemannian metric* on X .

From now on, it will be assumed that X is locally compact and Polish; i.e., the spaces Z_i are locally compact and Polish.

DEFINITION 5.1 (See [HH86], [CC00], [God91]). A foliated atlas $\mathcal{U} = \{U_i, \phi_i\}$ of \mathcal{F} is called *regular* if:

- (i) for all i , there is a foliated chart $(\tilde{U}_i, \tilde{\phi}_i)$ of \mathcal{F} so that $\overline{U_i} \subset \tilde{U}_i$ and $\tilde{\phi}_i|_{U_i} = \phi_i$;
- (ii) the cover $\{U_i\}$ of X is locally finite; and,
- (iii) for all i and j , the closure of every plaque of (U_i, ϕ_i) meets at most the closure of one plaque of (U_j, ϕ_j) .

Since X is Polish and locally compact, every foliated atlas $\mathcal{U} = \{U_i, \phi_i\}$ of (X, \mathcal{F}) is *refined* by a regular atlas $\mathcal{V} = \{V_\alpha, \psi_\alpha\}$ in the sense that, for each α , there is some index $i(\alpha)$ so that $\overline{V_\alpha} \subset U_{i(\alpha)}$ and $\phi_{i(\alpha)}$ extends ψ_α .

Let $\mathcal{U} = \{U_i, \phi_i\}$ be a regular foliated atlas of \mathcal{F} with $\phi_i : U_i \rightarrow B_i \times Z_i$, and let $p_i : U_i \rightarrow Z_i$ denote the composition of ϕ_i with the second factor projection $B_i \times Z_i \rightarrow Z_i$. Then the form (5.1) of the changes of coordinates holds globally, with $x_{ij} : Z_{ij} = \phi_j(U_i \cap U_j) \rightarrow \mathbb{R}^n$ and $h_{ij} : Z_{ij} \rightarrow Z_{ji}$. Each map h_{ij} is determined by the condition $p_j = h_{ij} p_i$ on $U_i \cap U_j$. They satisfy the cocycle property $h_{jk} h_{ij} = h_{ik}$ on $U_i \cap U_j \cap U_k$ for all i, j and k . The family $\{U_i, p_i, h_{ij}\}$ is called a *defining cocycle* of \mathcal{F} [Hae88], [Hae02].

The maps h_{ij} generate a pseudogroup \mathcal{H} of local transformations of $Z = \sqcup_i Z_i$, which is called the representative of the *holonomy pseudogroup* of \mathcal{F} induced by \mathcal{U} (or by $\{U_i, p_i, h_{ij}\}$). This \mathcal{H} is independent of \mathcal{U} up to pseudogroup equivalences. Let $E = \{h_{ij}\}$, which is a symmetric family of generators of \mathcal{H} . There is a canonical homomorphism between the leaf space and the orbit space, $X/\mathcal{F} \rightarrow Z/\mathcal{H}$, given by $L \mapsto \mathcal{H}(p_i(x))$ if $x \in L \cap U_i$.

By fixing any $b_i \in B_i$, each Z_i can be considered as a subset of X , called a *local transversal*, via

$$Z_i \equiv \{b_i\} \times Z_i \subset B_i \times Z_i \xrightarrow{\phi_i^{-1}} U_i.$$

²This means a section of the associated bundle over X of positive definite symmetric bilinear forms on the fibers of $T\mathcal{F}$, which is C^{m-1} as foliated map.

It can be assumed that all of these local transversals are mutually disjoint, and thus Z becomes embedded in X ; then it is called a *complete transversal* in the sense that it meets all leaves and is locally given by local transversals. Each \mathcal{H} -orbit injects into the corresponding \mathcal{F} -leaf in this way.

The *holonomy groups* of the leaves are the germ groups of the corresponding orbits. The leaves with trivial holonomy groups are called *leaves without holonomy*, and they correspond to orbits with trivial germ groups. Then the union X_0 of leaves with trivial holonomy groups is a dense G_δ saturated subset of X , hence Borel and residual (by Theorem 3.26, or see directly [Hec77a] and [EMT77]). When \mathcal{F} is transitive, the union $X_{0,d}$ of dense leaves with trivial holonomy groups is a residual subset of X (by Corollary 3.27).

By the regularity of \mathcal{U} , we can consider the foliated atlas $\tilde{\mathcal{U}} = \{\tilde{U}_i, \tilde{\phi}_i\}$ given by Definition 5.1-(i), where $\tilde{\phi}_i : \tilde{U}_i \rightarrow \tilde{B}_i \times \tilde{Z}_i$. By refining $\tilde{\mathcal{U}}$ if necessary, we can assume that it is also regular. Thus it also induces a representative $\tilde{\mathcal{H}}$ of the holonomy pseudogroup on $\tilde{Z} = \sqcup_i \tilde{Z}_i$, a symmetric set of generators $\tilde{E} = \{\tilde{h}_{ij}\}$ given by (5.1), and a defining cocycle $\{\tilde{U}_i, \tilde{p}_i, \tilde{h}_{ij}\}$. Observe that Z is an open subset of \tilde{Z} that meets all $\tilde{\mathcal{H}}$ -orbits, $\tilde{\mathcal{H}}|_Z = \mathcal{H}$, each \tilde{h}_{ij} extends h_{ij} , and $\overline{\text{dom } h_{ij}} \subset \text{dom } \tilde{h}_{ij}$.

Let (X', \mathcal{F}') be another locally compact Polish foliated space of class C^m and dimension n' . Then $\mathcal{F} \times \mathcal{F}'$ denotes the foliated structure on $X \times X'$ with leaves $L \times L'$, where L and L' are leaves of \mathcal{F} and \mathcal{F}' , respectively. Let $\mathcal{U}' = \{U'_\alpha, \phi'_\alpha\}$ be a foliated atlas of \mathcal{F}' , where $\phi'_\alpha : U'_\alpha \rightarrow B'_\alpha \times Z'_\alpha$. For all foliated charts $(U_i, \phi_i) \in \mathcal{U}$ and $(U'_\alpha, \phi'_\alpha) \in \mathcal{U}'$, we get a foliated chart $(U_i \times U'_\alpha, \psi_{i\alpha})$ of $\mathcal{F} \times \mathcal{F}'$, where $\psi_{i\alpha}$ is the composite

$$U_i \times U'_\alpha \xrightarrow{\phi_i \times \phi'_\alpha} B_i \times Z_i \times B'_\alpha \times Z'_\alpha \xrightarrow{\xi_{i\alpha}} B_{i\alpha} \times Z_i \times Z'_\alpha,$$

where $B_{i\alpha}$ is an open ball in $\mathbb{R}^{n+n'}$ and

$$\xi_{i\alpha}(x_i, y_i, x_j, y_j) = (\zeta_{i\alpha}(x_i, x_j), y_i, y_j)$$

for some homeomorphism $\zeta_{i\alpha} : B_i \times B'_\alpha \rightarrow B_{i\alpha}$. The collection $\mathcal{V} = \{U_i \times U'_\alpha, \psi_{i\alpha}\}$ is a foliated atlas of $\mathcal{F} \times \mathcal{F}'$. We can assume that the maps $\zeta_{i\alpha}$ are C^m diffeomorphisms, and therefore $\mathcal{F} \times \mathcal{F}'$ becomes of class C^m with \mathcal{V} .

Suppose that \mathcal{U}' is regular; in particular, it satisfies Definition 5.1-(i) with charts $(\tilde{U}'_\alpha, \tilde{\phi}'_\alpha)$, where $\tilde{\phi}'_\alpha : \tilde{U}'_\alpha \rightarrow \tilde{B}'_\alpha \times \tilde{Z}'_\alpha$. We can assume that each $\zeta_{i\alpha}$ extends to a homeomorphism $\tilde{\zeta}_{i\alpha} : \tilde{B}_i \times \tilde{B}'_\alpha \rightarrow \tilde{B}_{i\alpha}$ for some open balls $\tilde{B}_{i\alpha}$ containing $\overline{B_{i\alpha}}$. As before, using $\tilde{\phi}_i$, $\tilde{\phi}'_\alpha$ and $\tilde{\zeta}_{i\alpha}$, we get a foliated chart $(\tilde{U}_i \times \tilde{U}'_\alpha, \tilde{\psi}_{i\alpha})$ of $\mathcal{F} \times \mathcal{F}'$, which extends $\psi_{i\alpha}$. This shows that \mathcal{V} satisfies Definition 5.1-(i). The other conditions of Definition 5.1 are obviously satisfied, and therefore \mathcal{V} is regular. Observe that, if \mathcal{H}' is the representative of the holonomy pseudogroup of \mathcal{F}' induced by \mathcal{U}' , then $\mathcal{H} \times \mathcal{H}'$ is the representative of the holonomy pseudogroup of $\mathcal{F} \times \mathcal{F}'$ induced by \mathcal{V} .

5.2. Saturated sets

Consider the notation of Section 5.1. A subset of X is called *saturated* (or *\mathcal{F} -saturated*) if it is a union of leaves. The *saturation* (or *\mathcal{F} -saturation*) of a subset $A \subset X$ is the union $\mathcal{F}(A)$ of leaves that meet A . The canonical homeomorphism $X/\mathcal{F} \approx Z/\mathcal{H}$ gives a canonical bijection between the families of \mathcal{F} -saturated subsets of X and \mathcal{H} -saturated subsets of Z , which preserves the properties of being open, closed, G_δ , F_σ , Borel, Baire, dense, residual or meager.

REMARK 5.2. Note that each leaf is an F_σ subset of X . Moreover the relation set $R_{\mathcal{F}} \subset X \times X$ of the equivalence relation “being in the same \mathcal{F} -leaf” is an F_σ subset, which follows from Remark 3.21 since

$$R_{\mathcal{F}} = \bigcup_{i,j} (p_i \times p_j)^{-1} (R_{\mathcal{H}} \cap (Z_i \times Z_j)) ,$$

where each $p_i \times p_j$ is a trivial fiber bundle with σ -compact fibers.

The relation between saturations in X and Z is the following:

$$(5.2) \quad \mathcal{F}(A) = \bigcup_{i,j} p_j^{-1} (\mathcal{H}(p_i(A \cap U_i)) \cap Z_j)$$

for any $A \subset X$. Thus $\mathcal{F}(A)$ is open if A is open, which is well known. However the behavior of the saturation is worse in foliated spaces than in pseudogroups: the saturation of a meager, Borel or Baire set may not be meager, Borel or Baire, respectively. But we have the following result.

LEMMA 5.3. *Let $A \subset V \subset X$, where V is open in X . The following properties hold:*

- (i) *If A is residual in V , then $\mathcal{F}(A)$ is residual in $\mathcal{F}(V)$.*
- (ii) *If A is meager and $\mathcal{F}|_V$ -saturated in V , then $\mathcal{F}(A)$ is meager in $\mathcal{F}(V)$.*

PROOF. Property (i) follows from (5.2), Lemma 3.19 and Theorem 3.18 (applied to the trivial fiber bundles p_i).

To prove (ii), we can assume that V is some U_i . In this case, we have the following simplification of (5.2):

$$\mathcal{F}(A) = \bigcup_j p_j^{-1} (\mathcal{H}(p_i(A)) \cap Z_j) .$$

By Theorem 3.18, $p_i(A)$ is meager in Z_i . So $\mathcal{H}(p_i(A))$ is meager in Z by Lemma 3.20. Therefore each $p_j^{-1} (\mathcal{H}(p_i(A)) \cap Z_j)$ is meager in U_j by Theorem 3.18, obtaining that $\mathcal{F}(A)$ is meager. \square

5.3. Coarse quasi-isometry type of the leaves

Consider the notation of Sections 5.1 and 5.2, and suppose from now on that X is compact³. Let $R_{\mathcal{F}} \subset X \times X$ be the subset of pairs of points in the same leaf (the relation set of the partition into leaves). For $(x, y) \in R_{\mathcal{F}}$, let $d_{\mathcal{U}}(x, y)$ be the minimum number of \mathcal{U} -plaques whose union is connected and contains $\{x, y\}$. This defines a map⁴ $d_{\mathcal{U}} : R_{\mathcal{F}} \rightarrow [0, \infty)$, which is upper semi-continuous, symmetric and satisfies the triangle inequality, but it is not a metric⁵ on the leaves because $d_{\mathcal{U}} \geq 1$. To solve this problem, consider the function $d_{\mathcal{U}}^* : R_{\mathcal{F}} \rightarrow [0, \infty)$ given by

$$d_{\mathcal{U}}^*(x, y) = \begin{cases} d_{\mathcal{U}}(x, y) & \text{if } x \neq y \\ 0 & \text{if } x = y . \end{cases}$$

The map $d_{\mathcal{U}}^*$ is symmetric, satisfies the triangle inequality and its zero set is the diagonal $\Delta_X \subset R_{\mathcal{F}}$, and therefore its restriction to each leaf is a metric. However $d_{\mathcal{U}}^*$ is upper semi-continuous only on $R_{\mathcal{F}} \setminus \Delta_X$. For each $x \in X$, $S \subset L_x$ and $r \geq 0$,

³Several concepts of this section do not need compactness of X to be defined.

⁴The same definition gives a map $d_{\mathcal{U}} : X \times X \rightarrow [0, \infty]$ with the analogous properties so that $R_{\mathcal{F}}$ is the set with finite values.

⁵This $d_{\mathcal{U}}$ is a coarse metric on the leaves in the sense of Hurder [Hur94].

the notation $B_{\mathcal{U}}(x, r)$, $\overline{B}_{\mathcal{U}}(x, r)$ and $\text{Pen}_{\mathcal{U}}(S, r)$ will be used for the corresponding open and closed balls, and penumbra in $(L_x, d_{\mathcal{U}}^*)$.

A *plaque chain* (or *\mathcal{U} -plaque chain*) is a finite sequence of \mathcal{U} -plaques such that each pair of consecutive plaques has nonempty intersection. For $(x, y) \in R_{\mathcal{F}}$, the value $d_{\mathcal{U}}(x, y)$ equals the least $k \in \mathbb{Z}^+$ such that there is a plaque chain⁶ (P_1, \dots, P_k) with $x \in P_1$ and $y \in P_k$.

If X is C^1 , we can also pick up any Riemannian metric g on X and take the corresponding Riemannian distance d_g on the leaves, which defines an upper semicontinuous map $d_g : R_{\mathcal{F}} \rightarrow [0, \infty)$.

Since X is compact, \mathcal{U} is finite by Definition 5.1-(ii). Moreover $\overline{U_i}$ is compact, and therefore every Z_i has compact closure in \tilde{Z}_i . So Z has compact closure in \tilde{Z} . By the observations of Section 5.1, it follows that $\tilde{\mathcal{H}}$ is compactly generated and E is a symmetric system of compact generation of $\tilde{\mathcal{H}}$ on Z .

LEMMA 5.4. *E is recurrent.*

PROOF. There is an open cover $\{V_i\}$ of X such that $\overline{V_i} \subset U_i$ for all i . Then $W_i = p_i(V_i)$ is open with compact closure in Z_i . Thus $W = \bigcup_i W_i$ is open with compact closure in Z .

Take any $z \in Z$, which is in some Z_i . Then there is some $x \in U_i$ such that $p_i(x) = z$. Since $\{V_j\}$ covers X , there is also some V_j containing x . So $h_{ij}(z) = p_j(x) \in W_j$. This shows that $\mathcal{H}(z) \cap W$ is a 1-net in $\mathcal{H}(z)$ with d_E . \square

By Lemma 5.4 and Theorem 3.8, Z , \mathcal{H} and E satisfy the conditions to determine a coarse quasi-isometry type on the orbits that is “equi-invariant” by pseudogroup equivalences.

The following result is well known, at least in the case of foliations of manifolds. For the reader’s convenience, its proof is indicated by its relevance in this work, and because some subtleties show up in the case of foliated spaces.

PROPOSITION 5.5. *The following properties hold:*

- (i) *For any other regular foliated atlases \mathcal{V} of \mathcal{F} , $d_{\mathcal{U}}^*$ and $d_{\mathcal{V}}^*$ are equi-Lipschitz equivalent on the leaves.*
- (ii) *Suppose that \mathcal{F} is C^3 . Then, for any C^2 Riemannian metric g on X , $d_{\mathcal{U}}^*$ and d_g are equi-large scale Lipschitz equivalent on the leaves.*
- (iii) *The leaves with $d_{\mathcal{U}}^*$ are equi-coarsely quasi-isometric to the corresponding \mathcal{H} -orbits with d_E .*

PROOF. Let $R_i \subset R_{\mathcal{F}}$ be the relation set defined by the restriction $\mathcal{F}|_{U_i}$. Note that $\bigcup_i R_i$ is an open neighborhood of Δ_X in $R_{\mathcal{F}}$. By Definition 5.1-(i),(ii) and the compactness of X , it follows that each R_i has compact closure in $R_{\mathcal{F}}$. Hence, by the upper semicontinuity of $d_{\mathcal{V}}$ and since \mathcal{U} is finite, there is some $C > 0$ such that $\sup d_{\mathcal{V}}^*(R_i) \leq \sup d_{\mathcal{V}}(R_i) \leq C$ for all i . This means that the $d_{\mathcal{V}}^*$ -diameter of all plaques of \mathcal{U} is $\leq C$, obtaining that $d_{\mathcal{V}}^* \leq C d_{\mathcal{U}}^*$ on $R_{\mathcal{F}}$. This proves (i).

⁶For a leaf L , and plaques $P, Q \subset L$ (respectively, $x, y \in L$), let $\bar{d}_{\mathcal{U}}(P, Q)$ (respectively, $\bar{d}_{\mathcal{U}}(x, y)$) be the least $k \in \mathbb{N}$ such that there is a plaque chain (P_0, \dots, P_k) with $P_0 = P$ and $P_k = Q$. This defines a metric $\bar{d}_{\mathcal{U}}$ on the set of plaques in L , which can be identified to the metric d_E on the corresponding \mathcal{H} -orbit \mathcal{O} , where plaques in L are identified to the points in \mathcal{O} via the maps p_i . However the map $\bar{d}_{\mathcal{U}} = d_{\mathcal{U}} - 1 : L \times L \rightarrow [0, \infty)$, defined in this way, does not satisfy the triangle inequality.

Similarly, by the upper semicontinuity of d_g , we get $\sup d_g(R_i) \leq K$ for some $K > 0$, obtaining⁷ $d_g \leq Kd_{\mathcal{U}}^*$ on $R_{\mathcal{F}}$.

Consider the disjoint union of the leaves as a Riemannian manifold with g .

CLAIM 14. The disjoint union of the leaves has a positive injectivity radius.

For each i , take relatively compact open subsets, $Z'_i \subset Z_i$ and $B'_i \subset B_i$ such that the sets $U'_i = \phi^{-1}(B'_i \times Z'_i)$ cover X . Since \mathcal{U} is finite, it is enough to prove that, for all i , there is some $C_i > 0$ such that the injectivity radius of L_x at every $x \in U'_i$ is $\text{inj}_g(x) \geq C_i$. Let $\psi : \tilde{B}_i \rightarrow N$ be a C^3 open embedding into a closed n -manifold, and set $V = \psi(\tilde{B}_i)$ and $W = N \setminus \psi(\tilde{B}_i)$. Let $\{\lambda, \mu\}$ be a C^3 partition of unity of N subordinated to its open covering $\{V, W\}$. For all $z \in \overline{Z'_i}$, let g_z be the Riemannian metric on V that corresponds to g by the C^3 diffeomorphism

$$\tilde{\phi}_i^{-1}(\{z\} \times \tilde{B}_i) \xrightarrow{\tilde{\phi}_i} \{z\} \times \tilde{B}_i \equiv \tilde{B}_i \xrightarrow{\psi} V.$$

Pick up any C^2 Riemannian metric h on N . Then the metrics $h_z = \lambda g_z + \mu h$ ($z \in \overline{Z'_i}$) form a compact family of C^2 Riemannian metrics on N with the C^2 topology. By the continuous dependence of the injectivity radius on the Riemannian metric with respect to the C^2 topology on closed manifolds [Ehr74], [Sak83], there is some $K > 0$ such that $\text{inj}_{h_x}(y) \geq K$ for all $y \in N$. Let $K' > 0$ denote the infimum of the h_z -distance between $\psi(B'_i)$ and W , with z running in $\overline{Z'_i}$, and set $C_i = \min\{K, K'\} > 0$. Since $h_x = g_x$ on $\psi(B_i)$, we get $\text{inj}_g(x) \geq C_i$ for all $x \in U'_i$. This completes the proof of Claim 14.

By Claim 14, the continuous dependence of the geodesic flow on the metric with respect to the C^1 topology [Sak83, Lemma 1.5], and because X is compact, it easily follows that there is some $\varepsilon > 0$ such that

$$(5.3) \quad d_g^{-1}([0, \varepsilon)) \subset \bigcup_i R_i.$$

Take points x and y in a leaf L . Suppose first that $d_g(x, y) \geq \varepsilon/2$, and let $\gamma(t)$ ($0 \leq t \leq 1$) be a minimizing geodesic in L with $\gamma(0) = x$ and $\gamma(1) = y$ (L is a complete Riemannian manifold by Claim 14). Take a partition $0 = t_0 < t_1 < \dots < t_k = 1$ such that the length of $\gamma|_{[t_{l-1}, t_l]}$ is in $[\varepsilon/2, \varepsilon)$ for all $l \in \{1, \dots, k\}$. Observe that $d_g(x, y) \geq k\varepsilon/2$. By (5.3), for each $l \in \{1, \dots, k\}$, there is some index i_l such that $(\gamma(t_{l-1}), \gamma(t_l)) \in R_{i_l}$, obtaining

$$d_{\mathcal{U}}^*(x, y) = d_{\mathcal{U}}(x, y) \leq k \leq \frac{2}{\varepsilon} d_g(x, y).$$

Now, assume that $d_g(x, y) < \varepsilon/2$. Then (x, y) is in some R_i by (5.3), giving $d_{\mathcal{U}}(x, y) = 1$, and therefore $d_{\mathcal{U}}^*(x, y) \leq 1$. This shows that $d_{\mathcal{U}}^* \leq \frac{2}{\varepsilon} d_g + 1$ on $R_{\mathcal{F}}$, obtaining (ii).

Let L be an arbitrary \mathcal{F} -leaf, and \mathcal{O} the corresponding \mathcal{H} -orbit. Consider Z as a subset of X via the embedding of Z into X defined by an appropriate choice of the points b_i (Section 5.1). In this way, \mathcal{O} becomes a 1-net in L with $d_{\mathcal{U}}^*$. On the other hand, the \mathcal{U} -plaques can be identified to the points of Z via the maps p_i . Moreover, given two different \mathcal{U} -plaques, P of (U_i, ϕ_i) and Q of (U_j, ϕ_j) , we have $P \cap Q \neq \emptyset$ if and only if $p_i(P) \in \text{dom } h_{ij}$ and $h_{ij}p_i(P) = p_j(Q)$. Then, by using plaque chains, it easily follows that $d_E \leq d_{\mathcal{U}}^* \leq d_E + 1$ on the subset $\mathcal{O} \subset L$. Thus the

⁷This inequality only requires \mathcal{F} to be C^1 .

inclusion map $(\mathcal{O}, d_E) \rightarrow (L, d_{\mathcal{U}}^*)$ is a $(1, 1)$ -large scale bi-Lipschitz, and its image \mathcal{O} is a 1-net of $(L, d_{\mathcal{U}}^*)$. This gives (iii) by Lemma 2.22 and Proposition 2.27. \square

REMARK 5.6. The inclusion maps $(\mathcal{O}, d_E) \rightarrow (L, d_{\mathcal{U}}^*)$ of the proof of Proposition 5.5-(iii) are $(1, 3, 1)$ -large scale Lipschitz equivalences according to Lemma 2.22.

By Proposition 5.5-(i), when X is compact, the \mathcal{F} -leaves have a well determined coarse quasi-isometry type of metrics, represented by $d_{\mathcal{U}}^*$ for any regular atlas \mathcal{U} . Moreover, by Propositions 5.5-(ii) and 2.27, if X is C^3 , the quasi-isometry type of metrics on the leaves can be also represented by d_g for any C^2 Riemannian metric g on X .

All coarsely quasi-isometric invariants considered in the Chapter 1 make sense for the leaves with the above coarse quasi-isometry type. Via the identification of \mathcal{U} -plaques to points of Z (indicated in the proof of Theorem 5.5-(iii)), the definitions of growth and amenability of the \mathcal{F} -leaves given in Chapter 1, using \mathcal{U} -plaques, correspond to the definitions of growth and amenability as metric spaces with $d_{\mathcal{U}}^*$, and also to the growth and amenability of the \mathcal{H} -orbits with d_E .

Using the properties indicated in Sections 5.1 and 5.2, and Proposition 5.5-(iii), we get Theorems 1.2–1.3 and 1.6–1.8, 1.10 and 1.12 from Theorems 4.7, 4.8, 4.12–4.14, 4.17 and 4.21, applied to \mathcal{H} , because all of those theorems deal with (equi-) coarse quasi-isometric invariants. Similarly, Corollary 1.4 can be obtained from Corollary 4.9.

The subsets of $X \times X$ and $Z \times Z$ considered in Theorems 1.1 and 4.6 are obviously saturated by $\mathcal{F} \times \mathcal{F}$ and $\mathcal{H} \times \mathcal{H}$, respectively, and moreover they correspond one another by the canonical homeomorphism between the leaf space of $\mathcal{F} \times \mathcal{F}$ and the orbit space of $\mathcal{H} \times \mathcal{H}$. By Corollary 2.83, the subsets of $X \times X$ and $Z \times Z$ considered in Theorems 1.5 and 4.11 are also saturated by $\mathcal{F} \times \mathcal{F}$ and $\mathcal{H} \times \mathcal{H}$, respectively, and correspond one another. Hence, using the properties indicated in Sections 5.1 and 5.2, and Proposition 5.5-(iii), we also get Theorems 1.1 and 1.5 from Theorems 4.6 and 4.11.

FIRST PROOF OF THEOREM 1.13. Let B be a Baire subset of X such that the \mathcal{F} -saturation $\mathcal{F}(B)$ is not meager. For each $n \in \mathbb{N}$, let $B_n = \bigcup_L \text{Pen}_{\mathcal{U}}(L \cap B, n)$, where L runs in the family of \mathcal{F} -leaves. Since $B = B_0 \subset B_1 \subset \dots$ and $\bigcup_n B_n = \mathcal{F}(B)$ is not meager, there is some $N \in \mathbb{N}$ such that B_N is not meager. Then there is some nonempty open subset $V \subset X$ such that $B_N \cap V$ is residual in V . By refining \mathcal{U} if necessary, we can assume that this holds with $V = U_i$ for some $(U_i, \phi_i) \in \mathcal{U}$. By Theorem 3.18-(ii), $C = p_i(U_i \cap B_N)$ is residual in Z_i . Hence $\mathcal{H}(C) \cap Z_i$ is also residual in Z_i , and therefore $\mathcal{H}(C)$ is not meager in Z . By Theorem 3.28, it follows that there is some \mathcal{H} -saturated residual subset $Y \subset Z$ and some $R > 0$ such that $\mathcal{O} \cap C$ is an R -net in (\mathcal{O}, d_E) for all orbit $\mathcal{O} \subset Y$. Let Y' denote the \mathcal{F} -saturated residual subset of X that corresponds to Y . Let \mathcal{O} be an \mathcal{H} -orbit in Y and L the corresponding \mathcal{F} -leaf in Y' . Consider the embedding of Z into X for an appropriate choice of the points b_i (Section 5.1). In this way, according to the proof of Proposition 5.5-(iii), $\mathcal{O} \cap C$ is an $(R+1)$ -net in $(\mathcal{O}, d_{\mathcal{U}}^*)$, and \mathcal{O} a 1-net into $(L, d_{\mathcal{U}}^*)$. Thus $\mathcal{O} \cap C$ becomes an $(R+2)$ -net in L . Since

$$\mathcal{O} \cap C \subset \text{Pen}_{\mathcal{U}}(L \cap B_N, 1) \subset \text{Pen}_{\mathcal{U}}(L \cap B, N+1),$$

we get that $L \cap B$ is an $(R+3+N)$ -net in $(L, d_{\mathcal{U}}^*)$. \square

SECOND PROOF OF THEOREM 1.13. Consider first the case where B is open and non-empty. Then the following argument shows that the intersection of all leaves with B are equi-nets in the leaves. If this were wrong, then B would not cut a sequence of balls $B_{\mathcal{U}}(x_n, r_n)$ in the leaves with $r_n \rightarrow \infty$. Note that $\bigcap_m \overline{\bigcup_{n \geq m} B_{\mathcal{U}}(x_n, r_n)}$ is saturated because $r_n \rightarrow \infty$. Thus $X = \bigcap_m \overline{\bigcup_{n \geq m} B_{\mathcal{U}}(x_n, r_n)}$ by the minimality of \mathcal{F} , obtaining that the nonempty open set B cuts infinitely many balls $B_{\mathcal{U}}(x_n, r_n)$, a contradiction.

In the general case, with the notation of the first proof, there is some $N \in \mathbb{N}$ and a nonempty open subset $V \subset X$ such that $B_N \cap V$ is residual in V . By increasing N and reducing V if necessary, we can also suppose that $B_N \cap V$ is $\mathcal{F}|_V$ -saturated. Thus $V \setminus B_N$ is meager and $\mathcal{F}|_V$ -saturated. So $\mathcal{F}(V \setminus B_N)$ is meager by Lemma 5.3-(ii), and therefore the saturated set $Y = X \setminus \mathcal{F}(V \setminus B_N)$ is residual. Since $L \cap B_N \cap V = L \cap V$ for any leaf L in Y , it follows that the intersections of all leaves in Y with B_N are equi-nets in those leaves. Then the same property is satisfied with B like in the first proof. \square

5.4. Higson corona of the leaves

5.4.1. Higson compactification. Let L be a leaf of \mathcal{F} . To simplify the notation, let $L_{\mathcal{U}} = (L, d_{\mathcal{U}}^*)$, whose Higson compactification is denoted by $L_{\mathcal{U}}^{\nu}$. If \mathcal{V} is another regular foliated atlas of \mathcal{F} , then the identity map $L_{\mathcal{U}} \rightarrow L_{\mathcal{V}}$ is a large scale bi-Lipschitz bijection (Proposition 5.5-(i)). Therefore it induces a map $L_{\mathcal{U}}^{\nu} \rightarrow L_{\mathcal{V}}^{\nu}$, which is continuous at the points of νL and restricts to a homeomorphism between the corresponding coronas (Proposition 2.150-(i),(iii)). Thus the corona of $L_{\mathcal{U}}^{\nu}$ will be simply denoted by νL . The notation L^{ν} will be used for the underlying set of $L_{\mathcal{U}}^{\nu}$ equipped with the coarsest topology so that the identity map $L^{\nu} \rightarrow L_{\mathcal{U}}^{\nu}$ is continuous and the inclusion map $L \rightarrow L^{\nu}$ is an open embedding. The space L^{ν} is a compactification of L , called its *Higson compactification*, whose corona is νL . If \mathcal{F} is C^3 and g is a C^2 Riemannian metric on X , then the Higson compactification of (L, d_g) is L^{ν} .

Let \mathcal{O} be the \mathcal{H} -orbit that corresponds to L , equipped with d_E . Then \mathcal{O} becomes a subspace of both $L_{\mathcal{U}}$ and L with the injection $Z \rightarrow X$ given in Section 5.1. According to the proof of Proposition 5.5-(iii), the inclusion map $\mathcal{O} \rightarrow L_{\mathcal{U}}$ is $(1, 1)$ -large scale bi-Lipschitz and its image is a 1-net. Thus $\mathcal{O} \hookrightarrow L$ induces an embedding $\mathcal{O}^{\nu} \rightarrow L_{\mathcal{U}}^{\nu}$ (Corollary 2.152), which restricts to a homeomorphism $\nu \mathcal{O} \rightarrow \nu L$ (Remark 5.6 and Proposition 2.150-(iii)). If the above map $\mathcal{O}^{\nu} \rightarrow L_{\mathcal{U}}^{\nu}$ is considered as map $\mathcal{O} \rightarrow L^{\nu}$, then it is also an embedding because \mathcal{O} is subspace L . In this way, \mathcal{O}^{ν} becomes a subspace of both $L_{\mathcal{U}}^{\nu}$ and L^{ν} , with $\nu \mathcal{O} = \nu L$. Hence Theorem 1.11 is a direct consequence of Theorem 4.32.

Using Lemma 2.148, it easily follows that, for any compactification $\bar{L} \leq L^{\nu}$, there are unique compactifications, $\bar{L}_{\mathcal{U}} \leq L_{\mathcal{U}}^{\nu}$ and $\bar{\mathcal{O}} \leq \mathcal{O}_{\mathcal{U}}^{\nu}$, such that the identity map $L \rightarrow L_{\mathcal{U}}$ and inclusion map $\mathcal{O} \rightarrow L_{\mathcal{U}}^{\nu}$ have continuous extensions $\bar{L} \rightarrow \bar{L}_{\mathcal{U}}$ and $\bar{\mathcal{O}} \rightarrow \bar{L}_{\mathcal{U}}$ that restrict to the identity map on $\partial L = \partial \mathcal{O}$. In fact, $\bar{\mathcal{O}} = \text{Cl}_{\bar{L}_{\mathcal{U}}}(\mathcal{O}) = \text{Cl}_{\bar{L}}(\mathcal{O})$.

Consider the pseudogroup $\tilde{\mathcal{H}}$ on $\tilde{Z} = \bigsqcup_i \tilde{Z}_i$, with symmetric set of generators $\tilde{E} = \{\tilde{h}_{ij}\}$, induced by the foliated atlas $\tilde{\mathcal{U}}$ (Section 5.1). By refining $\tilde{\mathcal{U}}$ if necessary, we can assume that this foliated atlas is also regular. Then \tilde{E} is also a recurrent system of compact generation on \tilde{Z} of another compactly generated pseudogroup on

a larger space (Lemma 5.4). The points b_i also define a map $\tilde{Z} \rightarrow X$, which can be assumed to be the embedding. Thus we get the subspace inclusions $Z \subset \tilde{Z} \subset X$.

Let $\tilde{\mathcal{O}}$ be the $\tilde{\mathcal{H}}$ -orbit that corresponds to L , equipped with $d_{\tilde{\mathcal{E}}}$. As before, there are unique compactification $\overline{\tilde{\mathcal{O}}} \leq \tilde{\mathcal{O}}^\nu$, with corona $\partial\tilde{\mathcal{O}}$, where $\overline{\tilde{\mathcal{O}}} = \text{Cl}_{\tilde{L}}(\tilde{\mathcal{O}}) = \text{Cl}_{\tilde{L}_u}(\tilde{\mathcal{O}})$ and $\partial\tilde{\mathcal{O}} = \partial L$. Furthermore, by Proposition 3.9, the inclusion map $\mathcal{O} \rightarrow \tilde{\mathcal{O}}$ is bi-Lipschitz and its image is a net. Thus, as above, it induces an embedding $\overline{\mathcal{O}} \rightarrow \overline{\tilde{\mathcal{O}}}$, which restricts to a homeomorphism $\partial\mathcal{O} \rightarrow \partial\tilde{\mathcal{O}}$. In this way, we will consider $\overline{\mathcal{O}}$ as a subspace of $\overline{\tilde{\mathcal{O}}}$, with $\partial\mathcal{O} = \partial\tilde{\mathcal{O}}$; indeed, as explained before, $\overline{\mathcal{O}} = \text{Cl}_{\tilde{L}}(\mathcal{O}) \subset \text{Cl}_{\tilde{L}}(\tilde{\mathcal{O}}) = \overline{\tilde{\mathcal{O}}}$ with $\partial\mathcal{O} = \partial L = \partial\tilde{\mathcal{O}}$.

5.4.2. Limit sets. Let us continue with the notation of Section 5.4.1.

DEFINITION 5.7. The *limit set* of L at any $\mathbf{e} \in \partial L$, denoted by⁸ $\lim_{\mathbf{e}} L$, is the subset $\bigcap_V \text{Cl}_X(V \cap L)$ of X , where V runs in the collection of neighborhoods of \mathbf{e} in \overline{L} .

Like in the case of pseudogroups (Section 4.5.1), higher compactifications of the leaves induce smaller limit sets. Moreover $\lim_{\mathbf{e}} L$ is closed and nonempty⁹, which may not be \mathcal{F} -saturated. The following examples are foliated versions of Examples 4.23.

- EXAMPLES 5.8.** (i) For the one-point compactification L^* , the limit set of L at the unique point in the corona is the standard limit set of L , which is saturated.
(ii) For the compactification of L by the end space, we get the standard limit set of L at any end of L , which is also a saturated set.
(iii) Consider the set

$$\overline{L} = L \sqcup \text{Cl}_X(L) = (L \times \{0\}) \cup (\text{Cl}_X(L) \times \{1\})$$

with the topology determined as follows: $L \equiv L \times \{0\} \hookrightarrow \overline{L}$ is an open embedding of the leaf, and, a basic neighborhood of a point in $(x, 1) \in \text{Cl}_X(L) \times \{1\}$ in \overline{L} is of the form $(V \cap L) \sqcup V$, where V is any neighborhood of x in $\text{Cl}_X(L)$. This \overline{L} is a compactification of $L \equiv L \times \{0\}$. In terms of algebras of functions, \overline{L} corresponds to the algebra of \mathbb{C} -valued functions on L that admit a continuous extension to $\text{Cl}_X(L)$. The corona of \overline{L} is $\partial\overline{L} = \text{Cl}_X(L) \times \{1\} \equiv \text{Cl}_X(L)$. Moreover, for each $x \in \partial L$, it is easy to see that $\lim_x L = \{x\}$, which is not saturated if $\dim \mathcal{F} > 0$. (An analytic application of this compactification is given in [Can03].)

- (iv) For the Stone-Ćech compactification L^β , the limit set of L at any point in the corona L^β is a singleton by (iii).
(v) If $\overline{L} \leq L^\nu$, it will be shown that $\lim_{\mathbf{e}} L$ is \mathcal{F} -saturated for all $\mathbf{e} \in \partial L$ (Theorems 4.25 and 5.10).
(vi) As a particular case of (v), suppose that \mathcal{F} is C^∞ and g is a C^∞ Riemannian metric on X so that (L, g) has negative curvature. Then we can consider the

⁸When ∂L is the end space of L (Example 5.8-(ii)), it is standard to use the term \mathbf{e} -limit of L and the notation $\mathbf{e}\text{-}\lim L$. We prefer the stated terminology and notation because this concept represents the limit of the inclusion map $L \hookrightarrow X$ at \mathbf{e} (a formalization of a point at the “infinity” of L).

⁹If the compactness assumption on X is removed, then $\lim_{\mathbf{e}} L$ can be defined as well, but it may be empty.

compactification of L whose corona is the ideal boundary. The limit sets of L at points in its ideal boundary are \mathcal{F} -saturated.

For $S \subset X$ and $r \geq 0$, the *penumbra*¹⁰ of S of *radius* r is the set

$$\text{Pen}_{\mathcal{U}}(S, r) = \bigcup_{x \in S} \overline{B}_{\mathcal{U}}(x, r) .$$

Observe that $\text{Pen}_{\mathcal{U}}(S, 1)$ is the union of the \mathcal{U} -plaques that meet S .

LEMMA 5.9. *For all $\mathbf{e} \in \partial L$,*

$$\text{Pen}_{\mathcal{U}}(\lim_{\mathbf{e}} \mathcal{O}, 1) \subset \lim_{\mathbf{e}} L \subset \text{Pen}_{\tilde{\mathcal{U}}}(\text{Pen}_{\tilde{E}}(\lim_{\mathbf{e}} \tilde{\mathcal{O}}, 1), 1) .$$

PROOF. Since the inclusion map $\mathcal{O} \rightarrow L_{\mathcal{U}}$ is $(1, 1)$ -large scale bi-Lipschitz and its image is a 1-net, it is a $(1, 3, 1)$ -large scale Lipschitz equivalence (Remark 5.6). Thus, by Proposition 2.157, given a base \mathcal{V} of neighborhoods of \bar{e} in \mathcal{O}^{ν} , the sets $V' := \text{Cl}_{L_{\mathcal{U}}^{\nu}}(\text{Pen}_{\mathcal{U}}(V \cap \mathcal{O}, 1))$, with $V \in \mathcal{V}$, form a base \mathcal{V}' of neighborhoods of \mathbf{e} in L^{ν} . Since \mathcal{O} is a net in $L_{\mathcal{U}}$ and each \mathcal{U} -plaque is closed in $L_{\mathcal{U}}$, it easily follows that $\text{Pen}_{L_{\mathcal{U}}}(V \cap \mathcal{O}, 1)$ is closed in $L_{\mathcal{U}}$, and therefore

$$(5.4) \quad V' \cap L = \text{Pen}_{\mathcal{U}}(V \cap \mathcal{O}, 1) .$$

CLAIM 15. *For all $V \in \mathcal{V}$,*

$$\text{Cl}_X(V' \cap L) \subset \text{Pen}_{\tilde{\mathcal{U}}}(\text{Cl}_{\tilde{Z}}(\text{Pen}_E(V \cap \mathcal{O}, 1)), 1) .$$

So any $x \in \text{Cl}_X(V' \cap L)$ is the limit in X of some sequence $x_k \in V' \cap L$. By (5.4), there are sequences of points $y_k \in V \cap \mathcal{O}$, and indices i_k and j_k , such that $x_k \in U_{i_k}$, $y_k \in U_{i_k} \cap \mathcal{O} \cap Z_{j_k}$, and $p_{i_k}(x_k) = p_{i_k}(y_k) := z_k \in Z_{i_k}$ for all k . Then $z_k = h_{j_k i_k}(y_k) \in \text{Pen}_E(V \cap \mathcal{O}, 1)$. Since \mathcal{U} is finite, each U_i is relatively compact in \tilde{U}_i , and each Z_i is a relatively compact in \tilde{Z}_i , by passing to a subsequence if necessary, we can assume that there is an index i , and points $y \in \tilde{U}_i$ and $z \in \tilde{Z}_i$ such that $i_k = i$ for all k , and $y_k \rightarrow y$ in \tilde{U}_i and $z_k \rightarrow z$ in \tilde{Z}_i as $k \rightarrow \infty$. Thus $y \in \text{Cl}_{\tilde{Z}}(\text{Pen}_E(V \cap \mathcal{O}, 1))$. Moreover $x \in \tilde{U}_i$ and $\tilde{p}_i(x) = \lim_k \tilde{p}_i(x_k) = \lim_k z_k = z$, obtaining that $d_{\tilde{\mathcal{U}}}^*(x, y) \leq 1$. This shows Claim 15.

CLAIM 16. *For all $V \in \mathcal{V}$,*

$$\text{Pen}_{\mathcal{U}}(\text{Cl}_Z(V \cap \mathcal{O}), 1) \subset \text{Cl}_X(V' \cap L) .$$

For $x \in \text{Pen}_{\mathcal{U}}(\text{Cl}_Z(V \cap \mathcal{O}), 1)$, there is some $y \in \text{Cl}_Z(V \cap \mathcal{O})$ and some index i such that $x, y \in U_i$ and $p_i(x) = p_i(y) =: z \in Z_i$. Moreover, for some j so that $y \in Z_j$, there is some sequence $y_k \in V \cap \mathcal{O} \cap U_i \cap Z_j$ such that $y_k \rightarrow y$ in Z_j as $k \rightarrow \infty$. Since $p_i(x) = z = p_i(y) = \lim_k p_i(y_k)$, there is some sequence $x_k \in U_i$ such that $p_i(x_k) = p_i(y_k)$ and $x_k \rightarrow x$ as $k \rightarrow \infty$. Thus $d_{\mathcal{U}}(x_k, y_k) = 1$, giving $x_k \in V' \cap L$ by (5.4), and therefore $x \in \text{Cl}_X(V' \cap L)$. This proves Claim 16.

Now the result follows from Claims 15 and 16 by taking intersections with V running in \mathcal{V} . \square

The following is a more explicit version of Theorem 1.16.

¹⁰We can consider $d_{\mathcal{U}}$ as a “metric with possible infinite” values on X , defining $d_{\mathcal{U}}(x, y) = \infty$ when $L_x \neq L_y$. Then this definition of penumbra is the direct extension of the above one to “metrics with possible infinite values.”

THEOREM 5.10. *Let (X, \mathcal{F}) be a compact Polish foliated space, let \bar{L} be a compactification of an \mathcal{F} -leaf L , with corona ∂L , let \mathcal{H} be the representative of the holonomy pseudogroup induced by a regular foliated atlas, and let \mathcal{O} be the \mathcal{H} -orbit that corresponds to L . Suppose that $\bar{L} \leq L^\nu$, and let $\bar{\mathcal{O}} \leq \mathcal{O}^\nu$ be the compactification of \mathcal{O} that corresponds to \bar{L} , with corona $\partial \mathcal{O} = \partial L$. Then $\lim_{\mathbf{e}} L = \mathcal{F}(\lim_{\mathbf{e}} \mathcal{O})$ for all $\mathbf{e} \in \partial L$.*

PROOF. From Lemma 5.9, Theorem 4.25 and Proposition 4.30, we get

$$\mathcal{F}(\lim_{\mathbf{e}} \mathcal{O}) \subset \lim_{\mathbf{e}} L \subset \mathcal{F}(\lim_{\mathbf{e}} \tilde{\mathcal{O}}) = \mathcal{F}(\tilde{\mathcal{H}}(\lim_{\mathbf{e}} \mathcal{O})) = \mathcal{F}(\lim_{\mathbf{e}} \mathcal{O}). \quad \square$$

Now, Theorems 1.15 and 1.16 follow from Theorems 4.28, 4.29 and 5.10.

5.5. Versions with harmonic measures

Suppose that \mathcal{F} is differentiable of class C^2 , let g be a C^1 leafwise Riemannian metric on X , and let Δ denote the Laplacian defined by g on the leaves, mapping C^2 functions on X to continuous functions. A Borel measure μ on X is called *harmonic* if $\mu(\Delta f) = 0$ for all C^2 function f on X . This concept was introduced by Garnett [Gar83], who proved that they satisfy several relevant properties (see also [Can03], [CC03, Chapter 2]). For instance, there exists some harmonic probability measure on X . A harmonic measure on X is called *ergodic* (or *\mathcal{F} -ergodic*) if every \mathcal{F} -saturated set is either of zero measure or of full measure. Any \mathcal{H} -invariant measure on Z (a transverse invariant measure of \mathcal{F}) induces a harmonic measure on X . On the other hand, any harmonic measure μ on X induces an \mathcal{H} -invariant measure class $[\nu]$ on Z so that the \mathcal{F} -saturated sets of μ -zero measure correspond to \mathcal{H} -saturated sets of $[\nu]$ -zero measure, and $[\nu]$ is \mathcal{H} -ergodic if and only μ is \mathcal{F} -ergodic.

The following results follow directly from Theorems 4.34–4.38.

THEOREM 5.11. *Let (X, \mathcal{F}) be a compact Polish foliated space. With respect to an ergodic harmonic measure on X supported in X_0 ,*

- (i) *either almost all \mathcal{F} -leaves are coarsely quasi-isometric to almost all \mathcal{F} -leaves;*
- (ii) *or else almost all \mathcal{F} -leaves are coarsely quasi-isometric to almost no \mathcal{F} -leaf.*

THEOREM 5.12. *Let (X, \mathcal{F}) be a compact Polish foliated space. With respect to an ergodic harmonic measure on X supported in X_0 ,*

- (i) *either almost all \mathcal{F} -leaves have the same growth type;*
- (ii) *or else the growth type of almost all \mathcal{F} -leaves are comparable with the growth type of almost no \mathcal{F} -leaf.*

THEOREM 5.13. *Let (X, \mathcal{F}) be a compact Polish foliated space. With respect to an ergodic harmonic measure on X supported in X_0 , the equalities and inequalities of Theorem 1.8 are satisfied almost everywhere with some $a_1, a_3 \in [1, \infty]$, $a_2, a_4 \in [0, \infty)$ and $p \geq 1$.*

COROLLARY 5.14. *Let (X, \mathcal{F}) be a compact Polish foliated space. With respect to an ergodic harmonic measure on X supported in X_0 , any of the sets of Corollary 1.9 is either of zero measure or of full measure.*

THEOREM 5.15. *Let (X, \mathcal{F}) be a compact Polish foliated space. With respect to an ergodic harmonic measure on X supported in X_0 , almost all leaves have the same asymptotic dimension.*

5.6. There is no measure theoretic version of recurrence

We show that there is no measure theoretic version of Theorem 1.13. It fails for the most simple non-trivial minimal foliation: a minimal Kronecker flow on the 2-torus. Let us first prove the measure theoretic version of its pseudogroup counterpart (Theorem 3.28) fails for the pseudogroup generated by a rotation with dense orbits on the unit circle.

Let h be a rotation of the unit circle $S^1 \subset \mathbb{C} \equiv \mathbb{R}^2$ with dense orbits. Consider the action of \mathbb{Z} on S^1 induced by h (the action of each $n \in \mathbb{Z}$ is given by h^n). Consider the standard Riemannian metric on S^1 , and let Λ be the corresponding Riemannian measure. Thus the action is isometric and Λ is invariant. Moreover Λ is ergodic [Den32].

Let \mathcal{H} be the minimal pseudogroup on S^1 generated by the above action, which satisfies the conditions of Theorem 3.28, taking $U = S^1$, $\mathcal{G} = \mathcal{H}$ and $E = \{h, h^{-1}\}$. For each positive integer n , let I_n be an open arc in S^1 with $\Lambda(I_n) < \frac{1}{(2n+1)2^n}$. Then

$$A = \bigcup_{n=1}^{\infty} \bigcup_{i=-n}^n h^i(I_n)$$

is a Borel set with

$$\Lambda(A) \leq \sum_{n=1}^{\infty} \sum_{i=-n}^n \Lambda(h^i(I_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < 2\pi = \Lambda(S^1).$$

So its complement $B = S^1 \setminus A$ is a Borel set with $\Lambda(B) > 0$, and thus $\Lambda(\mathcal{H}(B)) = \Lambda(S^1)$ because Λ is ergodic. Nevertheless, every orbit \mathcal{O} meets each I_n at some point x , and thus

$$(5.5) \quad \mathcal{O} \cap A \supset \{h^i(x) \mid -n \leq i \leq n\} = \overline{B}_E(x, n).$$

Hence $\mathcal{O} \cap B$ is not a net in (\mathcal{O}, d_E) for any orbit \mathcal{O} .

The suspension of the above action produces a minimal Kronecker flow on the 2-torus. Equip \mathbb{R} with the standard Riemannian metric. Then the universal cover $\mathbb{R} \rightarrow S^1$, $t \mapsto e^{2\pi ti}$, is a local isometry. Let $\tilde{X} = \mathbb{R} \times S^1$, \tilde{g} the product Riemannian metric on \tilde{X} , and $\tilde{\mathcal{F}}$ the C^∞ foliation on \tilde{X} whose leaves are the fibers $\mathbb{R} \times \{x\}$ for $x \in S^1$. The Riemannian measure $\tilde{\mu}$ of \tilde{g} is harmonic for $\tilde{\mathcal{F}}$ with respect to the restriction of \tilde{g} to the leaves. Consider the C^∞ diagonal \mathbb{Z} -action on \tilde{X} , given by $n \cdot (t, x) = (n + t, h^n(x))$, which is isometric and preserves $\tilde{\mathcal{F}}$. Moreover $X = \mathbb{Z} \backslash \tilde{X}$ is a C^∞ manifold so that the quotient map $p : \tilde{X} \rightarrow X$ is a C^∞ covering map. Thus \tilde{g} and $\tilde{\mathcal{F}}$ project to a Riemannian metric g and a C^∞ foliation \mathcal{F} on X , and the Riemannian measure μ of g is harmonic for \mathcal{F} with respect to the restriction of g to the leaves. Note that \mathcal{H} is a representative of the holonomy pseudogroup of \mathcal{F} . Observe that the restriction $p : [0, 1/2] \times B \rightarrow p([0, 1/2] \times B) =: B'$ is bijective. So

$$\mu(B') = \tilde{\mu}([0, 1/2] \times B) = \frac{\Lambda(B)}{2} > 0.$$

On the other hand, suppose that there is an \mathcal{F} -leaf L so that $L \cap B'$ is a K -net in L for some $K \in \mathbb{N}$. Since \mathcal{H} is minimal, there is some $x \in I_{K+1}$ such that $L = p(\mathbb{R} \times \{x\})$. Then $L \cap B' = p((\mathbb{R} \times \{x\}) \cap p^{-1}(B'))$. Since the restriction $p : \tilde{L} \rightarrow L$ is an isometry with the restrictions of \tilde{g} and g , it follows that

$$(\mathbb{R} \times \{x\}) \cap p^{-1}(B') = \bigcup_{h^{-i}(x) \in B} [i, i + 1/2]$$

is a K -net in \mathbb{R} . But, by (5.5), $h^i(x) \notin B$ for $-K-1 \leq i \leq K+1$ because $x \in I_{K+1}$, and therefore 0 is not in the K -penumbra of $\bigcup_{h^{-i}(x) \in B} [i, i+1/2]$ in \mathbb{R} , a contradiction. Thus $L \cap B'$ is not a net in L for all \mathcal{F} -leaf L .

5.7. Open problems

Consider the notation and general conditions of Chapter 1.

PROBLEM 1. Prove versions of Theorems 1.1–1.3 for the relation “ $x \sim y$ if and only if L_x is differentiably quasi-isometric to L_y ” on X , assuming that \mathcal{F} is differentiable.

Problem 1 should not be difficult: the usual local Reeb stability and Arzela-Ascoli theorems should be enough to adapt the proofs of Theorems 1.1–1.3.

PROBLEM 2. Suppose that \mathcal{F} is minimal and residually many leaves have a Cantor space of ends.

- (i) Describe the possible coarse quasi-isometry types of the leaves in X_0 . Is it possible to describe also the coarse quasi-isometry types of the leaves with holonomy?
- (ii) If \mathcal{F} is also differentiable and of dimension 2, classify the possible differentiable quasi-isometry types of the leaves in X_0 .

Is it possible to do the same for the leaves with holonomy?

As indicated in Chapter 1, the version of Problem 2 for 2-ended leaves was solved by Blanc [Bla03]. In the case of a Cantor space of ends, one should try to prove that the leaves in X_0 are coarsely quasi-isometric to free group with two generators; as a motivation, recall that all finitely generated free groups with at least two generators are coarsely quasi-isometric (see e.g. [GdlH91, Example 10.11]). As a first step, it may be helpful to assume that all leaves in X_0 are equi-coarsely quasi-isometric one another (the alternative (i) of Theorem 1.2), or its differentiable version. We may even ask what can be said in the case of 1-ended leaves, but it seems to be much more difficult.

For an equivalence relation on a Polish space, several degrees of complexity were defined in the literature. One of them is the classification by countable models of the relation. Contrary to this condition, the relation may be generically ergodic with respect to the isomorphism relation on countable models. These properties are well understood in the case of equivalence relations defined by Polish actions [Hjo00], [Hjo02]. Some of the techniques used in that case were generalized in [ALC], and may be useful to address the following problem.

PROBLEM 3. Suppose that (X, \mathcal{F}) is transitive (or minimal), and satisfies the alternative (ii) of Theorem 1.2. Consider the following equivalence relations on X :

- (i) “ $x \sim y$ if and only if L_x is coarsely quasi-isometric to L_y .”
- (ii) “ $x \sim y$ if and only if L_x has the same growth type as L_y .”
- (iii) “ $x \sim y$ if and only if L_x is differentiably quasi-isometric to L_y ” (assuming that \mathcal{F} is differentiable).

Is any of these relations generically ergodic with respect to the isomorphism relation on countable models? Is there any example where some of them is classifiable by countable models?

As suggested by Hector, Problem 3 is especially interesting in the case of foliations of codimension one, confronting it with their Poincaré-Bedixson theory [HH87], [CC00].

PROBLEM 4. What can be said about their coarse cohomology and other coarse algebraic invariants [Roe93]?

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